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Periodic solutions of the restricted three-body problem for a small mass ratio☆

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Abstract

A plane circular restricted three body problem is considered for small values of the ratio of the masses μ of the main bodies. All the limit problems as $\mu \to 0$: the two-body problem, Hill's problem, the intermediate Hénon problem and the basic limit problem, are found using a Power Geometry. In each of them, solutions are isolated which are the limits of the periodic solutions of the restricted problem as $\mu \to 0$ and the limits of the families of periodic solutions (which are called generating families). Using the generating families in the case of small $\mu > 0$, the families are studied which are started as the reverse (family h) and forward (family i) circular orbits of infinitesimal radius around the body of greater mass. It is shown that, as μ increases, there is a small change in the structure of family h but family i undergoes infinitely many self-bifurcations with the formation of an infinite number of closed subfamilies, each of which only exists in a certain range of values of μ. A theory of the formation of horseshoe-shaped orbits and orbits in the form of "tadpoles" is given, and the structure of the basic families containing periodic solution with these orbits is indicated.

1. Introduction

1.1. Formulation of the problem

Suppose three point bodies P_1, P_2 and P_3 move in a single plane under the action of Newton's law of gravitation. The bodies P_1 and P_2 have masses m and m_2 respectively while the mass of the body P_3 is so small that its effect on the bodies P_1 and P_2 can be neglected. We shall say that the mass of the body P_3 is equal to zero. Then, the body P_1 executes a Kepler motion with respect to the body P_1. If the body P_2 moves along a circle, the problem of the motion of the body P_3 is referred to as a circular restricted three-body problem or, briefly, as the restricted problem. It was formulated for the first time by Euler.[1]

We shall assume that the units of mass, time and distance are chosen such that the sum $m_1 + m_2$, the gravitational constant, the distance P_1P_2 and the angular velocity of the body P_2 with respect to the body P_1 are equal to unity. The single parameter will then be $\mu = m_2 \in [0, 1/2]$. If a system of coordinates which rotates together with the body P_2 is now introduced, then, in this (synodic) system of coordinates with its centre at P_1, the position x_1, x_2 of the body P_3

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is described by a Hamiltonian system with two degrees of freedom and a single parameter μ (see [Ref. 2, Ch.3, §1])

$$\dot{x}_j = \partial H/\partial y_j, \quad \dot{y}_j = -\partial H/\partial x_j, \quad j = 1, 2 \tag{1.1}$$

where

$$H = H_0 + \mu R, \quad H_0 = (y_1^2 + y_2^2)/2 + x_2 y_1 - x_1 y_2 - r^{-1}$$
$$R = r^{-1} + x_1 - r_2^{-1}, \quad r = \sqrt{x_1^2 + x_2^2}, \quad r_2 = \sqrt{(x_1 - 1)^2 + x_2^2} \tag{1.2}$$

A derivative with respect to t is denoted by a dot.

When $\mu \neq 0$, the problem is not integrable in a finite form. The families of periodic solutions are of the greatest interest as they form a kind of a skeleton of a certain part of the phase space. For a fixed value of the parameter $\mu \neq 0$, the periodic solutions of the Hamiltonian system (1.1) form single-parameter families and, in the case of a variable μ, two-parameter families.[3]

For example, the periodic solutions for the following values of μ (the bodies $P_1 - P_2 - P_3$ are shown in brackets):

3.5×10^{-9} (Saturn - Janus (1980S1) - Erimetheus (1980S3));[4,5]
6.7×10^{-6} (Saturn - Mimas - a particle of Saturn's ring);[6]
5.178×10^{-5} (Sun - Neptune - a body of the Kuiper belt);[7–9]
9.538×10^{-4} (Sun - Jupiter - asteroid);[10–14]
1.215×10^{-2} (Earth - Moon - spacecraft).[15]

In addition, periodic solutions were calculated for other small values of μ and, also, for large values: $\mu = 0.4$ (Ref. 16) and $\mu = 0.5$ (Ref. 17) (in relation to the dynamics of particles and planets in the field of a double star).

It is not possible here to list all the papers on this theme (there are hundreds of them); we mainly mention those papers which are directly associated with the subject of this paper.

The majority of the papers are concerned with calculating of the families of periodic solutions for fixed values of μ where no developed theory is required. However, there is another approach: to consider limit problems as $\mu \to 0$ and their regular and singular perturbations for $\mu > 0$. This enables one to construct a theory of the families of periodic solutions for small μ.[18,2,11–14,19–22] This theory will be briefly described here and the results obtained from it will be compared with numerical results.

1.2. The contents of this paper

Four limit problems, which exist as $\mu \to 0$, are formulated in Section 2 and the periodic solutions of two of them (the problem of the two bodies P_2 and P_3 and Hill's problem) are considered.

The limit problem, that is, system (1.1), (1.2) when $\mu = 0$, is considered in Section 3. It is integrable, and all of its solutions can be described as it was done earlier in Ref. 2,Ch. III - VI. The phase space of this problem when $\mu = 0$ is complicated on account of collisions between the body P_3 and the body P_2, as a result of which the arc-solutions are formed. When $\mu > 0$, these collisions induce singular perturbations that leads to a further complication of the structure of the phase space.

The generating solutions, which are the limits of the periodic solution as $\mu \to 0$, are isolated out from the periodic solutions and families of arc-solutions of the principal limit problem. They form generating families, which are considered in Section 4. For solutions with perturbations which are regular with respect to μ, the separation of the generating families and the study of the generated families (that is, their perturbations for $\mu > 0$) is carried out using a normal form.[2] For solutions with singular perturbations, for which a collision between the bodies P_2 and P_3 occurs, the separation of the generating families is based either on the generalized Broucke principle[11–14,20] or on the theory of singular perturbations.[21]

Two generating families are considered as examples in Sections 5 and 6: one of them is simply organized and changes slightly as μ increases from 0 to 1/2, while the other is organized in a more complicated manner and, when μ increases from zero, it undergoes an infinite number of self-bifurcations, and an infinite number of closed subsets bifurcate from it, which only exist in small ranges of μ values.

A theory of perturbations for periodic orbits having the shape of “horseshoes” and “tadpoles” (these are traditional names) is constructed in Section 7. The structure and mutual arrangement of the main families containing these solutions are investigated.

Sections 5 and 6 have been jointly written while the remaining sections were written by the first author. The results of the Subsections 6.2–6.4 and Section 7 are being published for the first time.

1.3. *The principal properties of system (1.1) and its periodic solutions*[23,2]

The Orbit is the projection of the solution $x_j(t)$, $y_j(t)$ ($j = 1, 2$) of system (1.1) onto the plane x_1, x_2. If two families of periodic solutions intersect and the periods in one family are q times greater than the periods in the other family, we shall say that the first family has a (*local*) *multiplicity* q.

System (1.1) is transformed into itself under the substitution

$$t, x_1, x_2, y_1, y_2 \to -t, x_1, -x_2, -y_1, y_2 \tag{1.3}$$

which is its *symmetry*. Under the symmetry (1.3), the plane $x_2 = y_1 = 0$ is invariant and is called (Ref. 2, Ch. III) the *plane of symmetry*. The solutions of system (1.1) which are transformed into themselves under substitution (1.3) are *symmetric* solutions. A symmetric periodic solution intersects the plane of symmetry twice, and their family intersects II along two curves which are the characteristics of the family. It is convenient to track the mutual positions of the solutions along these intersections.

The family of periodic solutions of system (1.1) for a fixed value of the parameter μ is said to be *natural* if it is continued on both sides up to the natural ends, which can be a termination at a fixed point or in another family of periodic solutions, the contraction of the orbit into a point or its departure to infinity, the tendening of the period to zero or infinity, and, finally, a natural family can be a closed family.

The general properties of families of periodic solutions of a Hamiltonian system with two degrees of freedom have been presented in detail (Ref. 3,§ 1–3) and briefly (Ref. 2, Ch. II, § 4). For information on families of symmetric periodic solutions, see Ref. 3, § 5. Each symmetric solution M of a family F has a period T, a trace Tr of the monodromy matrix of the system in variations and two points of intersection with the symmetry plane II (over a half period). The values of the Hamiltonian function and the coordinates of the intersections with the plane of symmetry may serve as a parameter of the family.

In system (1.1) when $\mu \in (0, 1/2]$, the families of symmetric periodic solutions are two-parameter families and they can therefore have singularities with the codimension 1 and 2; several of them have been studied previously.[2,3] However, when $\mu = 0$, system (1.1) is degenerate.

2. Limit problems

2.1. *Derivation of the limit problems (Refs. 24–26, Ch. IV, § 4,27,28, § 1)*

In order to find all of the first approximations of the restricted three-body problem close to the body P_2 in the case of small μ, it is necessary to introduce the local coordinates

$$\xi_1 = x_1 - 1, \quad \xi_2 = x_2, \quad \eta_1 = y_1, \quad \eta_2 = y_2 - 1$$

and to expand the Hamiltonian function in these coordinates. After the expansion of $1/\sqrt{(\xi_1 + 1)^2 + \xi_2^2}$ in a MacLaurin series, the Hamiltonian (1.2) takes the form

$$\begin{aligned} H + \frac{3}{2} - 2\mu = \frac{1}{2}(\eta_1^2 + \eta_2^2) + \xi_2\eta_1 - \xi_1\eta_2 - \xi_1^2 + \frac{1}{2}\xi_2^2 + \\ + f(\xi_1, \xi_2^2) + \mu\left\{\xi_1^2 - \frac{1}{2}\xi_2^2 - \frac{1}{\sqrt{\xi_1^2 + \xi_2^2}} - f(\xi_1, \xi_2^2)\right\} \end{aligned} \tag{2.1}$$

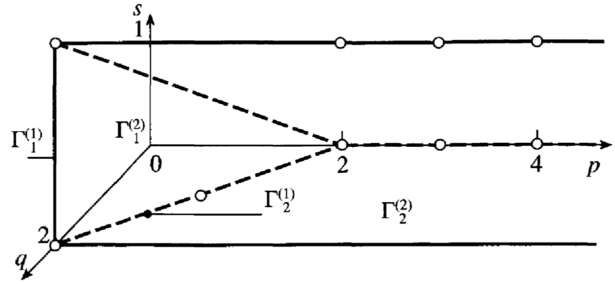


Fig. 1.

where f is a convergent power series which does not contain terms of order less than three. It is known[26] that the support $\mathbf{S}_1$ of the series on the right-hand side of equality (2.1) consists of the points

$$R=(r_1,r_2,r_3,r_4,r_5)=(\text{ord}\xi_1,\text{ord}\xi_2,\text{ord}\eta_1,\text{ord}\eta_2,\text{ord}\mu)\colon (0,0,2,0,0), (0,0,0,2,0), (0,1,1,0,0),$$
$$(1,0,1,0,0), (2,0,0,0,0), (0,2,0,0,0), (k,2l,0,0,0), (2,0,0,0,1), (0,2,0,0,1), (-1,0,0,0,1),$$
$$(0,-1,0,0,1), (k,2l,0,0,1),$$

where $k, l \geq 0$, $k+2l \geq 3$, and of the segment J joining the points $(-1, 0, 0, 0, 1)$ and $(0, -1, 0, 0, 1)$. This segment is the support of the term $\mu/\sqrt{\xi_1^2+\xi_2^2}$. The cone of the problem[26] is

$$\mathbf{K} = \{W \in \mathbb{R}_*^5 \colon w_1 < 0,\ w_2 < 0,\ w_5 < 0\}$$

as $\xi_1, \xi_2, \mu \to 0$.

We now make the projection

$$\pi R = R'' \overset{\text{def}}{=} (p, q, s) \in \mathbb{R}^3$$

where

$$p = r_1 + r_2 = \text{ord}\xi_i, \quad q = r_3 + r_4 = \text{ord}\eta_i, \quad s = r_5 = \text{ord}\mu$$

The set $\mathbf{S}''_1$ of these points R'' consists of

$$(0,2,0), (1,1,0), (2,0,0), (k,0,0), (2,0,1), (-1,0,1), (k,0,1),\ k=3,4,5,\ldots$$

The closure of the convex hull of the set $\mathbf{S}''_1$ is the polyhedron $\Gamma \subset \mathbb{R}^3$. The surface of the polyhedron $\mathbf{\Gamma}$ consists of the faces $\Gamma_j^{(2)}$, the edges $\Gamma_j^{(1)}$ and the vertices $\Gamma_j^{(0)}$. A truncated Hamiltonian function $\hat{H}_j^{(d)}$, which is the sum of those terms of series (2.1), the points of which R'' belong to $\Gamma_j^{(d)}$, corresponds to each such element $\Gamma_j^{(0)}$. The truncated Hamiltonian functions $\hat{H}_j^{(d)}$ are different first approximations of the function (2.1), which hold in different domains of space $(\xi_1, \xi_2, \eta_1, \eta_2, \mu)$. The cone of the problem is

$$\mathbf{K}'' = \{W'' \in \mathbb{R}_*^3 \colon w_1 < 0,\ w_3 < 0\}$$

The polyhedron $\mathbf{\Gamma}$ is a semi-infinite trihedral prism with a slanting base (Fig. 1). It has four faces and six edges.

The *face* $\Gamma_1^{(2)}$ serves as the slanting base of the prism $\mathbf{\Gamma}$. It contains the vertices $(0, 2, 0)$, $(2, 0, 0)$, $(-1, 0, 1)$ and the point $(1, 1, 0)$. Its normal vector $N''_1 = -(1, 1, 3) \in \mathbf{K}''$. The truncated Hamiltonian function

$$\hat{H} = \hat{H}_1^{(2)} = \frac{1}{2}(\eta_1^2+\eta_2^2)+\xi_2\eta_1-\xi_1\eta_2-\xi_1^2+\frac{1}{2}\xi_2^2-\frac{\mu}{\sqrt{\xi_1^2+\xi_2^2}} \tag{2.2}$$

corresponds to it. The power transformation

$$\xi_i = \tilde{\xi}_i\mu^{1/3}, \quad \eta_i = \tilde{\eta}_i\mu^{1/3}, \quad i = 1, 2 \tag{2.3}$$

reduces the function (2.2) to the form (2.2), where $\mu = 1$ and all the ξ_i, η_i are replaced by $\tilde{\xi}_i$, $\tilde{\eta}_i$. The Hamiltonian system

$$\dot{\tilde{\xi}}_j = \partial\hat{H}/\partial\tilde{\eta}_j, \quad \dot{\tilde{\eta}}_j = -\partial\hat{H}/\partial\tilde{\xi}_j; \quad j = 1, 2 \tag{2.4}$$

describes *Hill's problem*,[29] which is nonintegrable.

The *face* $\Gamma_2^{(2)}$ contains the points (0, 2, 0), (1, 1, 0), (2, 0, 0) and (*k*, 0, 0). Its normal vector $N''_2 = -(0, 0, -1) \in \mathbf{K}''$. The truncated Hamiltonian function $\hat{H} = \hat{H}_2^{(2)}$, which is obtained from the function H (1.2) when $\mu = 0$, corresponds to it. We call problem (2.4) the *basic limit problem*.

The remaining two faces have the normal vectors (0, −1, 0) and (0, 1, 2), lying outside the cone of the problem $\mathbf{K}''$, and the corresponding truncations of the Hamiltonian function are therefore not of use.

We now consider the edges. Of the six edges, one is improper. It passes through the point (0, 2, 0) parallel to the vector (1, 0, 0). On three edges $q = 0$, that is, the truncated Hamiltonian function for them is independent of η_1 and η_2, and, in the case of the solutions of the corresponding Hamiltonian system, $\xi_1, \xi_2 = \text{const}$, that is, they are of no interest. Two edges remain.

The edge $\Gamma_1^{(1)}$ contains the points (0, 2, 0) and (−1, 0, 1) of the set $\mathbf{S}''_1$. The corresponding truncated Hamiltonian function

$$\hat{H} = \hat{H}_1^{(1)} = \frac{1}{2}(\eta_1^2 + \eta_2^2) - \frac{\mu}{\sqrt{\xi_1^2 + \xi_2^2}} \tag{2.5}$$

describes *the two-body problem involving* P_2 and P_3 in a fixed system of coordinates. The power transformation

$$\xi_j = \mu\tilde{\xi}_j, \quad \eta_j = \tilde{\eta}_j, \quad t = \mu\tilde{t}, \quad j = 1, 2 \tag{2.6}$$

transfer it the Hamiltonian system (2.4) with a Hamiltonian function of the form (2.5), where ξ_j, η_j, μ are replaced by $\tilde{\xi}_j$, $\tilde{\eta}_j$, 1 respectively.

Suppose $\Gamma_0^{(2)}$ is the face which passes through the points (0, 2, 0), (2, 0, 1), (−1, 0, 1). Its external normal $N''_0 = (0, 1, 2)$ and the vector of the normal to the face $\Gamma_1^{(2)}$ is $N''_1 = -(1, 1, 3)$. Since the edge $\Gamma_1^{(1)}$ is the intersection of the faces $\Gamma_0^{(2)}$ and $\Gamma_1^{(2)}$, its normal cone consists of the vectors $N = (n_1, n_2, n_3) = \alpha N''_0 + \beta N''_1$, where $\alpha, \beta > 0$ and, if $\alpha < 3\beta/2$ and $\beta > 0$, then $n_1/n_3 \in (1/3, \infty)$.

The *edge* $\Gamma_1^{(2)}$ contains the points (2, 0, 0), (1, 1, 0), (0, 2, 0) of the set $\mathbf{S}''_1$. The truncated Hamiltonian function (2.2) with $\mu = 0$ corresponds to it. This function describes an *intermediate problem* (between Hill's problem and two-body problems involving P_1 and P_3) which is integrable. This first approximation was introduced by Hénon.[30] Since the vector of the normal to the face $\Gamma_1^{(2)}$ is $N''_1 = -(1, 1, 3)$ and the vector of the normal to the face $\Gamma_2^{(2)}$ is $N''_2 = (0, 0, -1)$, the normal cone to the edge $\Gamma_2^{(1)}$ consists of the vectors $N = (n_1, n_2, n_3) = \alpha N''_1 + \beta N''_2$, where $\alpha, \beta > 0$, that is $n_1/n_3 \in (0, 1/3)$.

We put $\alpha = \beta = 1$ and then obtain the vector −(1, 1, 4) lying in the normal cone of the edge $\Gamma_2^{(1)}$. The power transformation

$$\xi_j = \mu^{1/4}\tilde{\xi}_j, \quad \eta_j = \mu^{1/4}\tilde{\eta}_j, \quad j = 1, 2 \tag{2.7}$$

corresponds to it.

In the coordinates $\tilde{\xi}j$, $\tilde{\eta}_j$ as $\mu \to 0$, we obtain the limit Hamiltonian function (2.2) with $\mu = 0$, where, instead of ξ_j, η_j, there are $\tilde{\xi}j$ and $\tilde{\eta}_j$ respectively, and system (2.4). So, for the vectors $N = (n_1, n_2, n_3)$ lying in the normal cone of the edge $\Gamma_1^{(1)}$ and the relation $\nu \overset{\text{def}}{=} n_1/n_3 \in (1/3, \infty)$ for the vectors N from the normal cone of the face $\Gamma_1^{(2)}$, we have $\nu = 1/3$; for the vectors N from the normal cone of the edge $\Gamma_2^{(1)}$, we have $\nu \in (0, 1/3)$, and for the vectors N from the normal cone of the edge $\Gamma_2^{(2)}$, we have $\nu = 0$. Hence, if $\sqrt{\xi_1^2 + \xi_2^2} = O(\mu^\nu)$, then, very close to the body P_2, that is, for $\nu > 1/3$, the two-body problem for P_2 and P_3 with the Hamiltonian function (2.5) will be the first approximation of the restricted problem with Hamiltonian function (2.1); when simply close, that is, for $\nu = 1/3$, it is Hill's problem with Hamiltonian function (2.2); further from the body P_2, that is, for $1/3 > \nu > 0$, it is the intermediate Hénon problem; and, remote from the body P_2, that is, for $\nu = 0$, it is the basic limit problem. Close to the body P_2, the periodic solutions of the restricted problem are perturbations both of the periodic solutions of all the above-mentioned four first approximations as well as of the results of the splicing of the hyperbolic orbits of the two-body problem for P_2 and P_3 with the arc-solutions of the basic limit problem or the intermediate problem. The periodic solutions of the intermediate problem have been used[31–35] as generating solutions in the search for the periodic quasisatellite orbits of the restricted problem.

So, in the neighbourhood of the body P_2, there are three different desingularizations (2.6), (2.3) and (2.7) (that is, changes of the coordinates resolving the singularity) corresponding to the edge $\Gamma_1^{(1)}$, the face $\Gamma_1^{(2)}$ and the edge $\Gamma_2^{(1)}$. Of these, only desingularization (2.3) was known in the case of Hill's problem, that is, for the face $\Gamma_1^{(2)}$.

We consider the three limit problems (the two-body problem for P_2 and P_3, Hill's problem and the basic problem) below. Hénon's intermediate problem is explicitly integrable. It has been quite extensively investigated in Refs. 30–35 and is not considered here.

2.2. The two-body problem for P_2 and P_3 is described by the Hamiltonian system

$$\dot{\xi}_j = \partial\hat{H}/\partial\eta_j, \quad \dot{\eta}_j = -\partial\hat{H}/\partial\xi_j; \quad j = 1, 2 \tag{2.8}$$

with Hamiltonian function (2.5), where $\mu = 1$. This is an integrable problem and its solution has been described in detail in many books (see Ref. 36, for example). Here, we merely note that it has two families of periodic solutions with circular orbits: the family f with retrograde motion and the family g with direct motion. There are also other periodic solutions. However, only the two families remain under the perturbation of the restricted problem, and the remaining solutions are destroyed (Ref. 2, Introduction, p. 8; 20, §5.6). In addition, we note the existence of hyperbolic flyby orbits of the flight of the body P_3 close to the body P_2. The two-body problem is considered in greater detail below in Subsection 3.2.

2.3. Hill's problem (Refs. 30,37,14, § 2)

Is described by system (2.8) with the Hamiltonian function H from (2.2), where $\mu = 1$. There are other derivations of Hill's problem [Ref. 23, Ch. 10].

System (2.2), (2.8) possesses two symmetries:

$$t, \xi_1, \xi_2, \eta_1, \eta_2 \to -t, -\xi_1, \xi_2, \eta_1, -\eta_2 \tag{2.9}$$

$$t, \xi_1, \xi_2, \eta_1, \eta_2 \to -t, \xi_1, -\xi_2, -\eta_1, \eta_2 \tag{2.10}$$

In particular, $\Pi = \{\xi_2 = \eta_1 = 0\}$ is the plane of symmetry (2.10). Only periodic solutions with the symmetry (2.10) are considered below. The family of such solutions has two characteristics in the II plane. System (2.2), (2.8) has a singularity when $\xi_1 = \xi_2 = 0$ and two fixed points $L_1 = (-3^{-1/3}, 0, 0, -3^{-1/3})$, $L_2 = (3^{-1/3}, 0, 0, 3^{-1/3})$. At these points, the matrix of the linearized system (2.8) has two real eigenvalues and two pure imaginary eigenvalues, and a single family of periodic solutions therefore originates from each point L_1 and L_2. Hill's problem (2.2), (2.8) is non-integrable and has been investigated numerically. Hénon[30,37] has described its family of periodic solution most fully and represented their characteristics in $-2\hat{H}$, $\xi_1(0)$ coordinates. These characteristics are shown in the II plane in ξ_1, η_2 coordinates in Fig. 2 (Fig. 2, which was published for the first time by A. D. Bruno,[38] was made using the detailed tables sent by Hénon in 1979 which also contained the coordinates of the second point of intersection of the solution with the II plane). In Fig. 2, each family of periodic solutions is represented by two characteristics. We now enumerate the main families.

The family a leaves from the point L_2 (Ref. 30, Table 2).
The family c leaves from the point L_1 (Ref. 30, Table 2).
The family f starts with the circular orbits around the point $\xi_1 = \xi_2 = 0$ with retrograde motion, that is, clockwise (Ref. 30, Table 3). Its solutions possess the two symmetries: (2.9) and (2.10).
The family g starts with the circular orbits around the point $\xi_1 = \xi_2 = 0$ with direct motion (Ref. 30, Table 4). Its solutions also possess the two symmetries: (2.9) and (2.10). It contains a critical solution M with $\xi_1 = 0.28350$ and

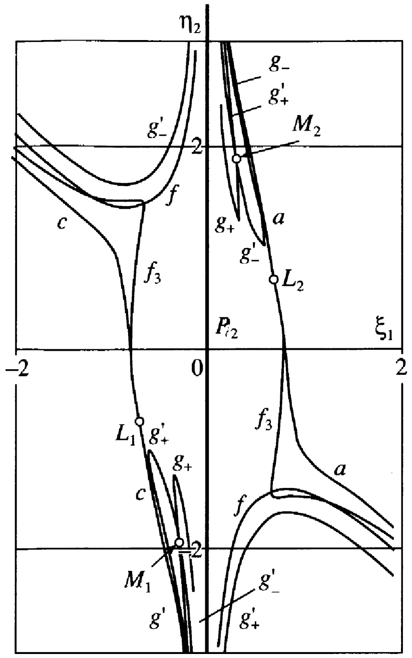


Fig. 2.

Table 1

k	$T/(2\pi)$	C	Tr	$\tilde{a}(0)$	$\tilde{e}(0)$	$\tilde{a}(T/2)$	$\tilde{e}(T/2)$
1	0.50	−1	[−2, +∞]	−1	−1	1	−1
2	1.50	2.679465	+∞	−1.47175	0.43384		∓∞
3	1.99	2.970940	[+∞, −∞]	−1.58720	0.63003	1.603	1.376
4	2.00	2.970934	[−∞, 2]	−1.58740	0.62996	1.587	1.370
5	2.00	0.629961	2	−1.58740	0	1.587	±2
6	2.00	−1.711013	[2, −∞]	−1.58740	−0.62996	1.587	−1,370
7	2.19	−1.785103	[−∞, +∞]	−1.76225	−0.53648	51.553	−2.019
8	2.50	−1.491531	+∞	−1.96669	−0.29891	0.511	−3.954
9	3.50	2.414539	+∞	−2.41232	0.23485		∓∞
10	3.99	2.929162	[+∞, −∞]	−2.51977	0.39686	2.539	1.606
11	4.00	2.929161	[−∞, 2]	−2.51984	0.39685	2.519	1.603
12	4.00	0.396850	2	−2.51984	0	2.519	±2
13	4.00	−2.135461	[2, −∞]	−2.51984	−0.39685	2.519	−1.603
14	4.05	−2.141393	[−∞, +∞]	−2.56117	−0.38821	5.877	−1.829
15	4.50	−1.645629	+∞	−2.82190	−0.19649	0.358	−4.790
16	5.50	2.312067	+∞	−3.20444	0.17058		∓∞

Tr = 2. In Fig. 2, the solution M is represented by the two points: M_1 and M_2. The solution M divides the family g into two parts: g_+ (Ref. 30, the upper part of Table 4, $a<1$) and g_- (Ref. 30, the lower part of Table 4, $a>1$).

The family g' intersects the family g at the solution M (Ref. 30, Table 5). The solution M subdivides the family g' into two parts: g'_- (Ref. 30, the second column of Table 5) and g'_+ (Ref. 30, the third column of Table 5 with all the minus signs replaced by plus signs). The parts g'_+ and g'_- turn into one another under the transformation (2.9).

The family f_3 is the family g_3 (Ref. 37, Table 1). It intersects the family f twice as (locally) a triple family. Its solutions possess two symmetries: (2.9) and (2.10).

Hénon[30] found the limits of the periodic solution of the family $a, \ldots, g'$ as $-2H \to +\infty$ and the other limit periodic solutions as solutions of the intermediate problem. By treating Hill's problem as a perturbation of this intermediate problem, Perko[39,40] proved, for large $-\hat{H}$, the existence of the family f and a denumerable number of families $g^{(n)}$,

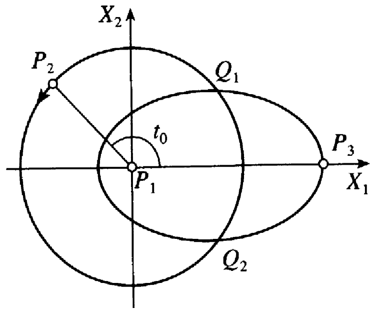


Fig. 3.

($n = 1, 2, \ldots$). However, this hypothesis regarding the arrangement of the families $g^{(n)}$ (Ref. 40, the end of Section 2 and in Fig. 3) is erroneous as the intersection all the families $g^{(n)}$ at the solution M is impossible. It is seen from this that the six families $a, \ldots, f_3$ which have been calculated are insufficient to describe the structure of all the families of periodic solution with the symmetry (2.10). It would be necessary to calculate further locally multiple families which intersect the family g' at the resonance solutions with $\mathrm{Tr} = -2, -1, 0$.

The restricted problem is a regular perturbation of Hill's problem with a small parameter $\mu^{1/3}$. Perko[41] began the study of perturbations for the families a, c, f, g, g', $g^{(n)}$. Since the restricted problem only has the symmetry (2.10) and does not have the symmetry (2.9), the latter symmetry is destroyed under the perturbation and, moreover, close to the solution M, at which the families g and g' intersect, bifurcation of these families must occur. The character of this bifurcation has not been specially studied but it is known due to calculations of the families of periodic solutions of the restricted problem.

In principle, all perturbations of the periodic solution of Hill's problem can be investigated using a normal form but this has still not been done.

3. The basic limit problem[2,18,20]

3.1. Introduction

The basic limit problem is problem (1.1), (1.2) with $\mu = 0$, that is, the two-body problem for P_1 and P_3 in a rotating system of coordinates for which, in the four dimensional phase space x_1, x_2, y_1, y_2, the plane

$$x_1 = 1, \quad x_2 = 0 \tag{3.1}$$

corresponding to the body P_2 is removed and, in order to describe its solution, we therefore initially consider the two-body problem for P_1 and P_3 in the fixed (sidereal) system of coordinates X_1, X_2, Y_1, Y_2 and, then, in the (synodic) system x_1, x_2, y_1, y_2, which rotates together with the body P_2:

$$(x_1 + ix_2)\exp(it) = X_1 + iX_2 \tag{3.2}$$

After this, we isolate out the new solutions which correspond to collisions between the body P_3 and the body P_2 and arise after the removal of plane (3.1) from the phase space. Then, in Section 4, we describe the *generating* solutions, that is, those solutions which are the limits of the solutions of problem (1.1), (1.2) as $\mu \to 0$. After this, in Subsection 4.3 and Sections 5 and 6, we consider examples of generating families of periodic solutions, that is, when $\mu = 0$ and the families which are generated by them when $\mu > 0$.

3.2. The two-body problem in a fixed system of coordinates (Ref. 2, Ch III; Ref. 36)

Suppose the body P_3 of zero mass moves in the plane X_1, X_2 under the action of the Newtonian attraction of the body P_1 of mass $m > 0$ which is at rest at the origin of coordinates $X_1 = X_2 = 0$. The motion of the body P_3 is described by the Hamiltonian system

$$\dot{X}_j = \partial h/Y_j, \quad \dot{Y}_j = -\partial h/X_j, \quad j = 1,2; \quad h = (Y_1^2+Y_2^2)/2 - m(X_1^2+X_2^2)^{-1/2} \tag{3.3}$$

In particular, $\dot{X}_j = Y_j$.

System (3.3) has three independent integrals: these are the energy integral and the space integrals

$$h = (Y_1^2+Y_2^2)/2 - m(X_1^2+X_2^2)^{-1/2} = -m(2a)^{-1}, \quad c = X_1Y_2 - X_2Y_1 \tag{3.4}$$

and $\tilde{\omega}$ is the length (angle) of the pericentre, that is, the points of the orbit with the smallest value of $(X_1^2+X_2^2)^{1/2}$.

The orbits in the plane X_1, X_2 are ellipses ($a>0$), parabolae ($a=0$) and hyperbolae ($a<0$) with a focus at zero: $X_1=X_2=0$, where the body P_1 is located. An elliptic orbit is uniquely defined by three parameters: the semi-major axis a, the eccentricity e and the length of the pericentre $\tilde{\omega}$. The magnitudes of a and e are related to integrals (3.4) by the equalities

$$a = a, \quad e = +\sqrt{1-c^2(am)^{-1}} \tag{3.5}$$

The position of the point P_3 in an orbit is given either by the true anomaly ν or by the mean anomaly l or the eccentric anomaly u. All the anomalies are angles measured from the direction to the pericentre such that, when the point P_3 passes across the pericentre $\nu=l=u=2\pi k$ and, when it passes across the apocentre, $\nu=l=u=2\pi k+\pi$, where k is an integer. The period of rotation T_s obeys Kepler law

$$(T_s/(2\pi))^2 = a^3 \tag{3.6}$$

The sign of the quantity c, which is determined by the last equality of (3.4), indicates the direction of motion along the ellipse: it is a direct motion when $c>0$ and a retrograde motion when $c<0$. We put $N = 2\pi T_s^{-1}$ and $n=N\operatorname{sgn}c$, and, then, n is the mean angular velocity of the motion of the point P_3 along the ellipse. According to inequality (3.6),

$$N = |n| = a^{-3/2} \tag{3.7}$$

If $e=0$ ($am=c^2$), the orbit is a circle of radius a and the quantity $\tilde{\omega}$ here loses its meaning. If $e=1$ ($c=0$), the orbit is a segment of length $2a$, one end of which is located at zero. The motion along the segment starts with the point P_3 leaving from the point P_1 and, after a time T_s, it terminates with the collision of these points. The direction of motion (the sign of n) loses its meaning here.

3.3. Synodic orbits

According to equality (3.2), in the rotating (synodic) coordinates x_1, x_2, the motion is described by the system of equations (1.1), where

$$H = (y_1^2+y_2^2)/2 + x_2y_1 - x_1y_2 - mr^{-1} \tag{3.8}$$

and integrals (3.4) take the form

$$h = (y_1^2+y_2^2)/2 - mr^{-1} = -m(2a)^{-1}, \quad c = x_1y_2 - x_2y_1 \tag{3.9}$$

On changing to the rotating system of coordinates, the conical sections are twisted into more complex orbits and only the circular obits ($e=0$) retain their shape. A synodic orbit, corresponding to elliptic motion, is confined in the annulus

$$a(1-e) \le r \le a(1+e) \tag{3.10}$$

The motion occurs with a mean angular velocity n - 1. If the number N is irrational, then the orbit is never closed and its points are everywhere dense in the above mentioned annulus. If N is rational, we put

$$N = |n| = (p+q)/p, \quad -N = -|n| = (p+q')/p, \quad q' = -(2p+q) \tag{3.11}$$

where $p>0$ and q are relatively prime numbers. So, in the case of rational N, a synodic orbit is closed after q revolutions about the origin of coordinates if $n>0$ and, after q' revolutions if $n<0$. The synodic period of such orbits is $T=2\pi p$.

All the synodic orbits with fixed n and e are obtained from one such orbit as it revolves about zero by same angle. A unique point of a sidereal orbit corresponds to each point of a synodic orbit. On the other hand, generically, several points of a synodic orbit correspond to a single point of a sidereal orbit and, in fact, $p+q$ points for a rational N and a denumerable family in the case of irrational N. The points of the minimum (maximum) polar radius r for a synodic orbit correspond to the pericentre (apicentre) of the sidereal orbit and, at these points $\dot{r}=0$.

In the four-dimensional space $\mathbb{R}^4$ of the variables x_1, x_2, y_1, y_2, the Hamiltonian function (3.8) is analytic everywhere apart from the plane $P_1^* \stackrel{\text{def}}{=} \{x_1=x_2=0\}$, which corresponds to the body P_1. In this domain $\mathcal{G}_0 \stackrel{\text{def}}{=} \mathbb{R}^4 \setminus P_1^*$, we isolate the set $\mathcal{D}$ by the condition $a>0$ (see equality (3.9)). The set $\mathcal{D}$ consists of all of those and only those trajectories the sidereal orbits of which are ellipses.

We will now consider the intersections of the trajectories from the set $\mathcal{D}$ with the plane of symmetry $\Pi=\{x_2=y_1=0\}$. In this plane, it is more convenient to introduce the new coordinates $\tilde{a}$, $\tilde{e}$:

$$\tilde{e} = x_1 y_2 |y_2| m^{-1}, \quad \tilde{a} = x_1 |2-|\tilde{e}||^{-1} \tag{3.12}$$

Then,

$$a = |\tilde{a}|, \quad e = |1-|\tilde{e}||, \quad n = \operatorname{sign}\tilde{e}\, a^{-3/2} \tag{3.13}$$

In the coordinates $\tilde{a}$, $\tilde{e}$, the intersection of the set $\mathcal{D}$ with the plane Π is the zone

$$\tilde{a} \in \mathbb{R}\backslash 0, \quad \tilde{e} \in [-2, 2] \tag{3.14}$$

The two straight lines $\tilde{e}=1$ and $\tilde{e}=-1$ correspond to circular orbits with a period T and a trace Tr:

$$T = 2\pi|n-1|^{-1}, \quad \mathrm{Tr} = 2\cos T \tag{3.15}$$

if $n \neq 1$. If $n=1$, the corresponding points $\tilde{a}=\pm 1$, $\tilde{e}=1$ are fixed. On the straight line $\tilde{e}=1$, the sidereal motion is direct (the set Id) and, on the straight line $\tilde{e}=-1$, it is a retrograde motion (the family Ir).

The four zones $\tilde{e} \in (-2, -1), (-1, 0), (0, 1), (1, 2)$ correspond to elliptic sidereal orbits with a direct motion if $\tilde{e}>0$ and retrograde motion if $\tilde{e}<0$. In the case of fixed $a=|\tilde{a}|$ with a rational $N=a^{-3/2}=(p+q)/p$, the corresponding synodic orbits are periodic with a period $T=2\pi p$ and a trace $\mathrm{Tr}=2$ (the families $E_N^{\pm}$). All the points of these zones either correspond to pericentres if $|\tilde{e}|>1$ or to apocentres if $|\tilde{e}|<1$.

The three straight lines $\tilde{e}=0, \pm 2$ correspond to solutions in which collisions occur between the body P_3 and the body P_1. Here, the straight lines $\tilde{e}=\pm 2$ can be considered as being coincident; they correspond to the points of collision.

3.4. *The restricted problem when $\mu=0$ (Ref. 2, Ch. III, § 3)*

Suppose the mass of the body P_2 is equal to zero. Although the body P_2 does not attract the body P_3, collisions between them are possible. This is the difference between the restricted problem when $\mu=0$ and the synodic two-body problem. The synodic motion of the body P_3 is now described by the same system (1.1) with the same Hamiltonian function H (3.8) but now in the domain $\mathcal{G}=\mathcal{G}_0 \setminus P_2^*$, where P_2^* is the plane $x_1=1, x_2=0$, corresponding to the body P_2 in phase space. The points of collision of the body P_3 with the body P_2 (that is, the points corresponding to the solution of the two-body problem lying in the plane P_2^* or, what is the same, the point $x_1=1, x_2=0$ in an orbit of the two-body problem) partition the solution of the two-body problem into pieces which will now no longer be a continuation of one another. Each such piece is an independent solution of the restricted three-body problem with $\mu=0$. The pieces which start and end at the points of collision are of particular interest. We shall call them arc-solutions or *solutions with successive collisions*. In the restricted problem, the arc-solutions play approximately the same role as periodic solutions.

The sidereal orbits of the body P_2 (the circle) and of the body P_3 (the ellipse) are shown in Fig. 3. Collisions can only occur at the points of intersection of these orbits Q_1 and Q_2. Suppose two successive collisions occur at times $t_1<t_2$.

Two cases are possible.

Case 1. The collision at t_2 occurs at the same point Q_1 of the sidereal plane as the collision at the time t_1. The bodies P_2 and P_3 then each make several revolutions in their own orbits between the two collisions. For this, their sidereal periods of rotation must be commensurate, that is, N is a rational number. An arc is a periodic trajectory with a deleted point. The set of such arcs is denoted by T_N.

Case 2. The collision at the time t_2 occurs at the other point Q_2 of the sidereal plane. The points Q_1 and Q_2 are symmetric about the X_1 axis (see Fig. 3). It follows from symmetry considerations that the bodies P_2 and P_3 are located on the X_1 axis at the time $(t_1+t_2)/2$. The corresponding synodic orbit is symmetric about the x_1 axis. The set of such arc-solutions is denoted by S.

Each arc-solution is found for the solution of the two-body problem for specific values of *a, e* and *c*. The set of arc-solutions (that is, solutions with successive collisions) consists of a denumerable set of the one-parameter families T_N existing for all rational $N < 2^{2/3} \approx 1.587$ and consisting of asymmetric arc-solutions and a denumerable set of single-parameter families S consisting of symmetric arc-solutions which also decompose into the families A_i, B_j, C_{kl}, where the integers $i \geq 0, j \geq 1, l \geq k \geq 1$. Hénon[18] found the families S for the first time and a theory of them families was then developed (Ref. 2, Ch. III–VI; Ref. 20). The structure of the families T_N has been studied (Ref. 2, Ch. III). Without dwelling on a statement of this theory, we merely note some of its implications.

In the Π plane and in the coordinates $\tilde{a}$, $\tilde{e}$, the curve

$$P_2^{**} = \{2 - |\tilde{e}| = 1/\tilde{a}\}$$

corresponds to the body P_2. It is represented by the dot-dash curve in Fig. 4. A collision is only possible within the annulus

$$1 - e \leq a^{-1} \leq 1 + e$$

In Fig. 4 (Fig. 4*a* is on the left-hand side and Fig. 4*b* is on the right-hand side), the domains ω_1, ω_2, ω_3, ω_4 bounded by the dashed curves and the curve P_2^{**} correspond to it. The characteristic curves of the families S are represented in these domains. Note that each symmetric arc-solution intersects the Π plane at a single point and their one-parameter family intersects along a curve which is called the characteristic of the family. The orbits of the arc-solutions A_i, B_j, C_{kl} have been presented in Ref. 2, Ch. IV.

4. Generating families of periodic solutions

4.1. Generating solutions

Suppose the periodic solution M_μ of the restricted problem (1.1), (1.2), which exists for a certain $\mu > 0$ can be continued continuously up to $\mu = 0$ and, in the limit, gives a solution (not a point) of one of the limit problems. This limit is called a *generating periodic solution.* According to Hénon (Ref. 20, § 2.10), there are three forms of generating periodic solutions depending on the limit M_0 in the main limit problem.

The first form. All points of the solution M_0 are separated from the body P_2.
The second form. The solution M_0 has at least one point on the body P_2 and one point outside the body P_2.
The third form. The solution M_0 lies as a whole in the body P_2.

A generating solution of the first form is a periodic solution of the synodic problem for the two bodies P_1 and P_3. A generating solution of the second form consists of several arc-solution of the basic limit problem. Since, when $\mu > 0$, the restricted problem has the integral H from (1.2), all the arc-solutions which are parts of generating periodic solution have the same value of the integral H or the Jacobi constant $C = -2H$. A generating periodic solution of the third form is a periodic solution of Hill's problem or the intermediate Hénon problem. The limit of the family of periodic solutions as $\mu \to 0$ is called the *generating family*. It can consist of generating periodic solutions of different forms.

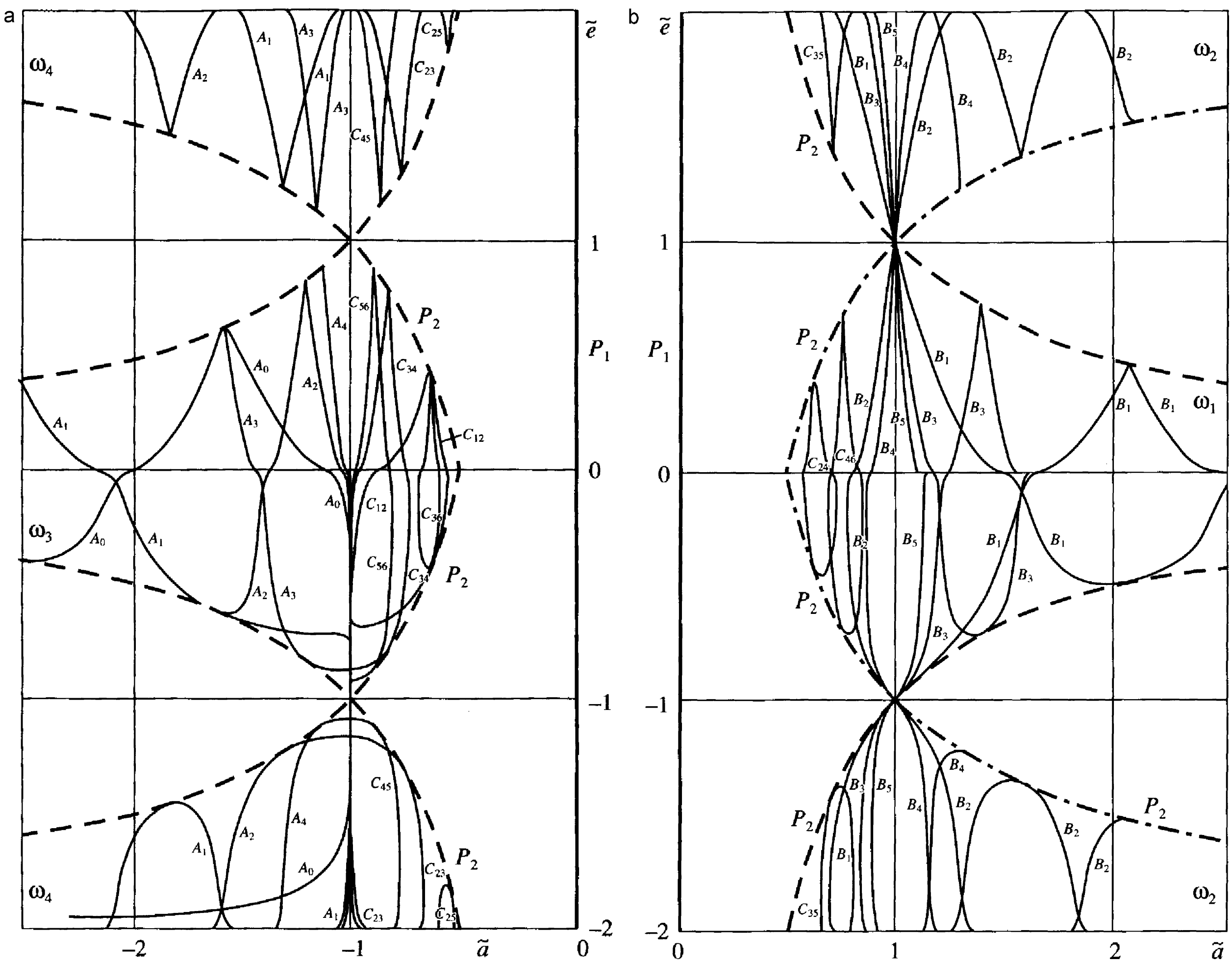


Fig. 4.

4.2. Generating solutions of the first form

All generating periodic solutions of the first form and their bifurcations have been investigated (Ref. 2, Ch. VII, VIII). All symmetric periodic solutions relate to them. They form two families Id and Ir with circular orbits and a denumerable number of families $E_N^{\pm}$ with elliptic orbits with a fixed semi-major axis (one or two families for each rational $N=a^{-3/2}>0$) (Ref. 2, Fig. 11). Bifurcation between these families occurs at sites of the intersection of the family Id with the families $E_{(p+1)/p}$ for $p=1, \pm 2, \pm 3, \ldots$. We denote these intersections by $\mathrm{Id}(N)=\mathrm{Id}\cap E_N$. In the plane II, points with $N=(p+1)/p$ for $p=1, \pm 2, \pm 3, \ldots$, that is, $\tilde{a}=\pm N^{-2/3}$, $\tilde{e}=1$ correspond to these intersections of the families. They divide the family Id in pieces Id_p with $p/(p-1)>N>(p+1)/p$ for $p=1, \pm 2, \pm 3, \ldots$ and the family $E_{(p+1)/p}$ into two parts: $E^{+}_{(p+1)/p}$ (with $\tilde{\omega}=0$) and $E^{-}_{(p+1)/p}$ (with $\tilde{\omega}=\pi$). The bifurcations of the pieces Id_p with the families $E_N^{\pm}$ are shown in Fig. 5. Perturbations of period T and trace Tr have been presented (Ref. 2, Table 2 Appendix) for certain families E_N.

In addition, there are families $G_{1/p}$ of asymmetric generating solutions for $p=1, 2, \ldots$. For them, $N=1/p$. The family G_1 originates from the fixed Lagrange point L_4 as a family of short-period solutions, intersects with the family $E^{-}_{1/p}$ and terminates at the fixed point L_5. The family $G_{1/p}$ with $p>1$ is closed, it intersects with the family $E^{-}_{1/p}$ twice and does not intersect with the family $E^{+}_{1/p}$.

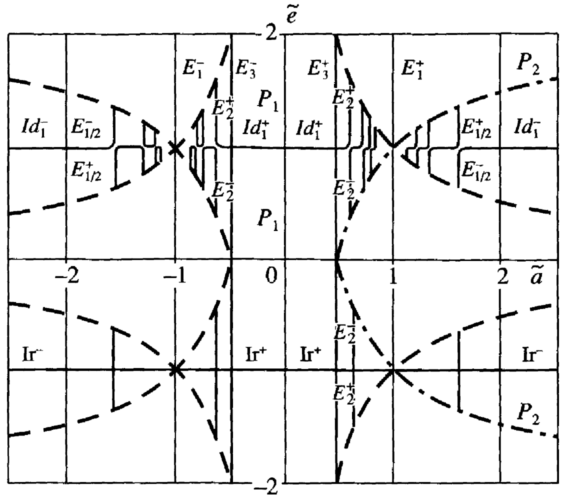


Fig. 5.

4.3. Generating solutions of the second form

When $\mu = 0$, every symmetric periodic solution with a collision of P_3 with P_2 is formed by arc-solutions from the families S and an even number of symmetrically arranged arcs from the families T_N, and the value of the Jacobi constant C is the same for all the arcs (Ref. 2, Ch. III, IV). Apparently, each such combination is generating. This, however, has only been proved for the simplest of them consisting of only one or two arc-solutions.[42] The families S, T_N and their characteristics in Π have been studied (Ref. 2, Ch. III–V). Bifurcations between the families of symmetric periodic solutions occur at the sites of intersection of the families Id, Ir, E_N^+ and S. For a rational $N = (p+q)/p$, $N \neq 1$, intersections of the family E_N with the families S occur in orbits for which

$$\tilde{e} = \psi(N, k),\ k = \pm 0, \pm 1, \ldots, \pm|p+q| \text{ when } N > 1 \text{ and } k = 0, 1, \ldots, 2|p+q| \text{ when } N < 1$$

where ψ is a certain function. Specific orbits (Ref. 2, Ch. VI, Sect. 3, Ch. IV, Theorem 2.4), which we denote by $E_N(k)$, correspond to these values of $\tilde{e}$.

There are two other orbits of intersection (Ref. 2, Ch. III, 3):

$E_1(-1) = \{N = 1, \tilde{e} = -1\}$ (here the families Ir, E_1^+, E_1^- and S intersect);
$E_1(1) = \{N = 1, \tilde{e} = 1\}$ (here the families Id, E_1^+ and S intersect).

So, in the orbits Id$(1 + 1/p)$ and $E_N(k)$, bifurcations occur between pieces of the generating families of the symmetric periodic solutions.

The extremal orbits (with respect to the Jacobi constant C) of the families S also a play considerable role in the formation of the generating families. A theory of them has been developed (Ref. 2, Ch. IV, V) and numerical values have been published.[43] For each family S of the type A_i and B_j, the initial extremal of an orbit $S(0)$ is selected and the direction of the increase in the numbering is such that the extremal orbits are denoted by $S(k)$, where the integer k can also be negative. In the case of the families A_i, the orbit $A_i(0)$ is a multiple of the orbit $E_1(-1)$, that is, $\tilde{a} = \pm 1, \tilde{e} = -1$. In the case of the family B_j, the orbit $B_j(0)$ is $E_1(1)$, that is, $\tilde{a} = 1, \tilde{e} = 1$. For the families C_{kl}, there are only two extremal orbits: $C_{kl}(1)$ when $\tilde{e} > 0$ and $C_{kl}(-1)$ when $\tilde{e} < 0$. Tables of the families S are available.[18]

The basic generating families Id, Ir, $E_N^{\pm}$, A_i, B_j, C_{kl} and T_N have been described (Ref. 2, Ch. III, IV) and, also, the points of intersection of their characteristics in the plane of symmetry II, which correspond to special solutions for which the basic families intersect:

$$\mathrm{Id}(N) = \mathrm{Id} \cap E_N, \quad \mathrm{Ir}(N) = \mathrm{Ir} \cap E_N, \quad E_N(k) \tag{4.1}$$

Table 2

k	$T/(2\pi)$	C	Tr	$\tilde{a}(0)$	$\tilde{e}(0)$	$\tilde{a}(T/2)$	$\tilde{e}(T/2)$
1	1/2	3.4668	−2	−0.4807	1	0.4807	1
2	1	3.1748	2	−0.6300	1	0.6300	1
3	1	1.5874	2	−0.6300	±2	0.6300	±2
4	1	0	2	−0.6300	−1	0.6300	−1
5_1	1	0.3027	[2, +∞]	−0.6300	−0.4126	0.6300	−0.4126
6_1	1	1.7935	+∞	−0.5575	0	0.6657	−0.4978
7_1	1	2.8720	[+∞, 2]	−0.6300	0.4126	0.6300	0.4126
8	1	3.1748	2	−0.6300	1	0.6300	1
9	3/2	3.0926	−2	−0.7114	1	0.7114	1
10	2	3.0575	2	−0.7631	1	0.7631	1
11_2	2	2.0876	[2, −∞], [−∞, +∞]	−0.7631	0.1044	0.7631	1.8956
12_2	2.281	2.8741	[+∞, −∞]	−0.6329	0.4182	0.9001	1.3695
13_2	2.056	1.7935	−∞	−0.5575	0	0.7377	1.9669
14_2	1.974	1.4018	−∞	−0.5548	−0.0368	0.7133	±2
15_2	2	−0.2960	[−∞, 2]	−0.7631	−0.6068	0.7631	−1.3932
16	2	−0.4367	2	−0.7631	−1	0.7631	−1
17_1	2	−0.3505	[2, +∞]	−0.7631	−1.3104	0.7631	−0.6896
18_1	2	1.4845	+∞	−0.6736	∓2	23.872	−1.9581
19_1	2	2.9712	[+∞, 2]	−0.7631	1.3104	0.7631	0.6896
20	2	3.0575	2	−0.7631	1	0.7631	1
21	5/2	3.0392	−2	−0.7991	1	0.7991	1
22	3	3.0285	2	−0.8255	1	0.8255	1
23_2	3	2.5673	[2, −∞], [−∞, +∞]	−0.8255	1.6657	0.8255	1.6657
24_2	3.347	2.9712	[+∞, −∞]	−0.7634	1.3099	0.9507	1.1765
25_2	3.088	1.4845	−∞	−0.6736	±2	0.7548	1.9214
26_2	2.990	1.4018	−∞	−0.6718	−1.9985	0.7133	±2
27_2	3	−0.5544	[−∞, 2]	−0.8255	−1.2358	0.8255	−1.2358
28	3	−0.6057	2	−0.8255	−1	0.8255	−1
29_1	3	−0.5646	[2, +∞]	−0.8255	−0.7886	0.8255	−0.7886
30_3	2.858	−0.2960	+∞	−0.7631	−0.6067	1.0894	−1.0821
31_3	2.973	1.4018	+∞	−0.5548	−0.0368	2.0701	−2.4830
32_3	3.111	1.7935	+∞	−0.5575	0	1.5137	−2.6606
33_3	3.575	2.8741	[+∞, −∞]	−0.6329	0.4182	1.1566	−1.1354
34_3	3.103	2.0876	[−∞, +∞]	−0.7631	0.1044	1.4578	−2.6859
35_1	3	2.9874	[+∞, 2]	−0.8255	0.7885	0.8255	0.7885
36	3	3.0285	2	−0.8255	1	0.8255	1
37	7/2	3.0216	−2	−0.8457	1	0.8457	1
38	4	3.0170	2	−0.8618	1	0.8618	1
39_2	4	2.7447	[2, −∞], [−∞, +∞]	−0.8618	0.4786	0.8618	1.5214

Moreover, the extremal orbits described above

$$A_i(k), \quad B_j(k), \quad C_{kl}(\pm 1) \tag{4.2}$$

in which regression (folding) or closure of the corresponding characteristic of the generating family is possible, have been isolated.[2] Bifurcation of the intersecting basic families occurs in the special solutions (4.1), that is, the generating family consists of pieces of the basic families bounded by the special solutions. The nature of these bifurcations has been discussed[20,21] but has not been so for studied in all cases.

The trace $\mathrm{Tr} = \pm\infty$ for the generating families of periodic solutions of the second form (that is, with collisions of the bodies P_3 and P_2). A change in the sign of Tr only occurs in special and extremal solutions for families which have periodic solutions consisting of a single arc-solution of the families S (that is, A_i, B_j or C_{kl}).[44] For families of periodic solutions consisting of more than a single arc-solution and, besides these, other sites where the sign of the trace Tr changes are also possible, but nothing is known about them at the present. However, in the case of a regular Hamiltonian system at the locus of the intersection of two families of periodic solutions, the trace $\mathrm{Tr} = 2$ for one family and $\mathrm{Tr} = 2\cos(2\pi/q)$ for the other family, where the natural number q is the order of the resonance and the multiplicity of the intersection. Single-piece generating families can therefore only intersect for the special solutions (4.1) and extremal solutions (4.2) while multipiece generating families can also intersect for as yet unknown solutions.

4.4. Examples of generating families [Refs. 14,20; Ch. 1]

The *family m* begins as a part of the family Ir with a from ∞ to 1 and with $\tilde{e} = -1$. When $a = 1$, it passes into a part of the family A_0 with $\tilde{e} < -1$, including segments of hyperbolic orbits. Here, Tr = −2 and $a = 1$ and Tr = $-\infty$ for the family A_0 from $a = 1$ up to the extremal orbit $A_0(-1)$.[43]

In the orbit $A_0(-1)$, the trace Tr jumps over from $-\infty$ to $+\infty$ and, subsequently, Tr = $+\infty$.

The family c begins from the fixed point L_1 as family c of Hill's problem and then passes into a part of the family B_1 with $a \le 1$ and $|\tilde{e}| \ge 1$ from $\tilde{e} = 1$ to $\tilde{e} = -1$ and $a = 1$, where it ends as a double family in the family h.

The family a starts from the fixed point L_2 as family a of Hills problem and then passes into a part of the family B_1 with $a \ge 1$ and $|\tilde{e}| \le 1$ from $\tilde{e} = 1$ to $E_{1/2}(t)$. Here, the family B_1 intersects the family A_0 and with another part of the family B_1. The family $B_1 + 2A_0$ subsequently proceeds. In this case, in the family A_0 the values of N change from ½ to 1 and the values of the Jacobi constant C decrease from -1 while, in the family B_1, the values of N decrease from ½ to a value N', which corresponds to C = -1. Suppose that, in the family B_1 when $N = N'$ and $C = -1$, we have $a = a'$, $\tilde{e} = \tilde{e}'$. Then, a piece of the family $B_1 + B_1$ proceeds from C = -1 up to the value $C = C''$ which corresponds to the extremal arc-solution $B_1(-1)$,[43] where $a = a''$ and $\tilde{e}^- = \tilde{e}''$. In this case, the value of C is the same for two orbits of the family B_1 but they belong to different parts of the family B_1: in one orbit $1 \le a \le a''$ and $-1 \le \tilde{e} \le \tilde{e}''$ and, in the other orbit, $a'' \le a \le a'$ and $\tilde{e}'' \le \tilde{e} \le \tilde{e}'$. The family finishes as a double family in the extremal orbit $B_1(-1)$.

The family b begins from the fixed point L_3 as the family E_1^- from $\tilde{e} = -1$ to $\tilde{e} = -1$ and terminates here as a double family in the family h.

The family f begins as the family f of retrograde circular orbits of the two-body problem for P_3 and P_2. It then passes into the family f of Hills problem and, then, into the family E_1^+ from $\tilde{e} = 1$ to $\tilde{e} = -1$. Here, it passes into a part of the family B_1 with $a \ge 1$ from $a = 1$, $\tilde{e} = -1$ to the orbit $E_{1/3}(0)$ and, subsequently, into the family $E_{1/3}^+$ up to the orbit $E_{1/3}(2)$, into the family B_1 up to the orbit $E_{1/5}(0)$, into the family $E_{1/5}^+$ up to the orbit $E_{1/5}(2)$, into the family B_1 up to the orbit $E_{1/7}(0)$, and so on.

The family g starts as the family g of the direct circular orbits of the two-body problem of P_3 and P_2. It then passes into the family g_+ of Hills problem. For the solution M, it passes into the family g'_+ of Hills problem. It then passes into the piece of the family B_2 from $a < 1$ up to $a = 1$, $\tilde{e} = -1$. Here, it passes into the family $T_1 + T_1$, each orbit of which is formed by two different orbits of the family T_1 which are symmetric to one another about the x_1 axis. This piece terminates when $\tilde{a} = 1$, $\tilde{e} = 1$ where it passes into the family g_- of Hills problem. For the solution M, it passes into the family g'_- of Hills problem which is continued from $\tilde{a} = 1$, $\tilde{e} = 1$ by the piece of the family B_2 with $a > 1$. Its further structure has been described (Ref. 20, Section 10.2.5) but the termination of the family is unknown.

The family l begins as part of the family Id with $a \in (\infty, 2^{2/3}]$, that is, with $N \in (0, 1/2]$. After the orbit Id(1/2), the family passes into the family $E_{1/2}^-$ up to the orbit of intersection $E_{1/2}(1)$, where it passes into the family $A_0 + B_1$.

Perturbations of the generating families are poorly understood. Only the shift of the trace Tr for the families E_N (Ref. 2, Fig. 74) and, also the perturbations of the trace Tr and period T in the families E_N (Ref. 2, Appendix, Table 2) are known. The perturbations of the generating families have been traced when $\mu \approx 10^{-3}$ (Ref. 14) and $\mu \approx 10^{-2}$ (Ref. 15).

We next consider the evolution of the two families h and i as μ increases from 0 to ½. We recall that an orbit is said to be *critical* if either Tr = ±2 in it or there is a collision of the body P_3 with the body P_1 or P_2 or the Jacobi constant has an extremum along the family in the case of fixed μ.

5. The family *h*

The family h starts with retrograde circular orbits of infinitesimal small radius about the body P_1 of larger mass.

5.1. The generating family h ($\mu = 0$)

The generating family h was initially described (Ref. 14, § 3) as the family IR+ and, subsequently, (Ref. 2, Section 10.2.6) as the family h.

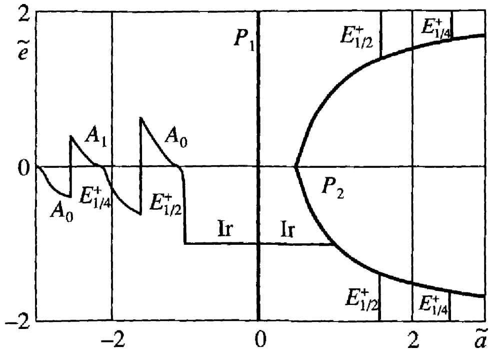


Fig. 6.

Data for 16 critical orbits of the calculated part of this family are shown in Table 1: the number of an orbit k, the normalized period of the orbit $T/(2\pi)$, the value of the Jacobi constant C, the trace Tr (or the interval of its change), the initial points of the orbit in the astronomical coordinates $\tilde{a}(0)$ and $\tilde{e}(0)$ according to relations (3.12) and the point of the orbit over a half period $\tilde{a}(T/2)$, (for $k=2$, 9, 16, the value of $\tilde{a}(T/2)$ is undefined).

The characteristics of the family in the coordinates $\tilde{a}$, $\tilde{e}$ are shown in Fig. 6. The generating family begins as part of the family Ir of retrograde circular orbits around the body P_1 of unit mass. This part is terminated by orbit 1, where the family h passes into the part of the family A_0 with $\tilde{e} > -1$ up to orbit 4. Here, a collision with the body P_2 occurs in orbit 2 and this collision persists when $\mu > 0$ and, in orbit 3, the Jacobi constant C reaches a maximum. In orbit 4, the family h becomes the family $E^+_{1/2}$ up to orbit 6. At the same time, there is a collision with the body P_1 in orbit 5. From orbit 6, the family h continues as the family A_1 up to orbit 11. Here, the Jacobi constant C reaches a minimum in orbit 9. From orbit 11, the family h continues as the family $E^+_{1/4}$ up to orbit 13. Here, a collision with the body P_1 occurs in orbit 12. From orbit 13, the family h continues as the family A_0 up to the intersection with the family $E^+_{1/6}$ while the Jacobi constant C reaches a minimum in orbit 14; a collision with the body P_2 occurs in orbit 16. On the whole, the family h consists of the pieces

$$\{A_0, E^+_{1/(4k-2)}, A_1, E^+_{1/(4k)}\}, \quad k = 1, 2, \ldots$$

The first such piece ($k=1$) and the beginning of the second piece ($k=2$) have been described above.

5.2. *Evolution of the family h as μ increases from 0 to 1/2*

For $\mu = 0.00095388$, which corresponds to the Sun - Jupiter case, the family h was calculated (Ref. 14, § 4) as the family $IR+J$ (see also Ref. 45, § 2 and Ref. 22, § 7). In the plane $\tilde{a}$, $\tilde{e}$, the characteristics of this family barely differ from the characteristics of the generating family shown in Fig. 6. The family h was calculated for $\mu = 0.1$ and $\mu = 0.2$,[45] and for $\mu = 0.3$, 0.4, 0.5.[46] The characteristics of the family h are shown in Fig. 7 when $\mu = 0.1$, 0.3 and 0.5. The evolution

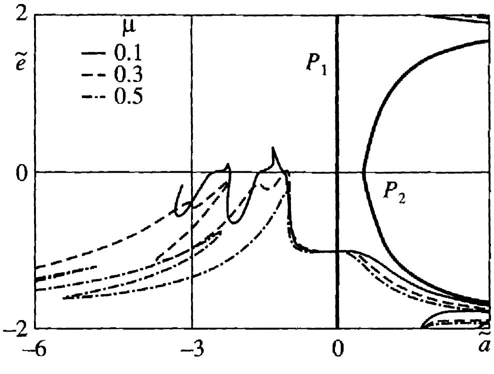


Fig. 7.

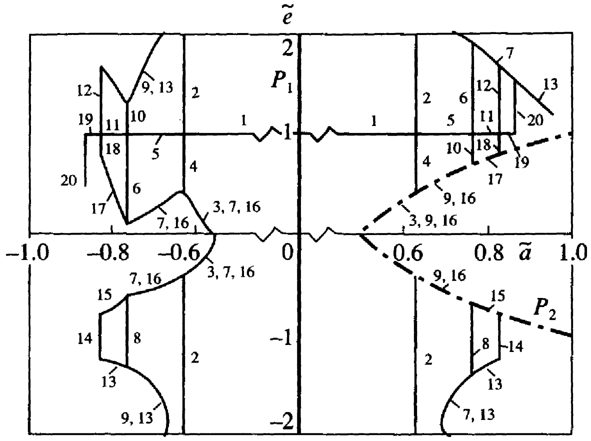


Fig. 8.

of the family h as μ increases can be seen and, in this case, no new singularities appear and self-bifurcations do not occur. For $\mu \approx 0.012$, the family h was calculated[15] as the family A_1.

6. The family *i*

The family i is started by direct circular orbits of infinitesimal small radius around the body P_1 of larger mass. Unlike the family h, described in Section 5 and which is simply arranged for all $\mu \in [0, 0.5]$, the family i has rather complicated structure.

6.1. The generating family i ($\mu = 0$)

The initial part has been successively described in (Ref. 11, § 20, Section 10.2.7 and Ref. 47, Section 1.1). The bifurcations of the family Id with the families $E_{(p+1)/p}$ have been described in Subsection 4.2.

Data on the initial critical orbits of this family, constructed in an analogous way to Table 1, are shown in Table 2, where the subscript m in the number k_m indicates the number of arc-solutions from which this orbit is constituted.

The characteristics of the family in the system of coordinates $\tilde{a}, \tilde{e}$ are shown in Fig. 8. The numbers l of pieces $\mathbf{K}_l$ from which the start of the generating family is composed are indicated. Some segments of the characteristics are identical: in the left characteristic $\mathbf{K}_3 \subset \mathbf{K}_7 \subset \mathbf{K}_{16}$ and $\mathbf{K}_9 \subset \mathbf{K}_{13}$, and in the right characteristic $\mathbf{K}_3 \subset \mathbf{K}_9 \subset \mathbf{K}_{16} \subset P_2^{**}$ and $\mathbf{K}_7 \subset \mathbf{K}_{13}$.

- $\mathbf{K}_1$. The family Id from $a = 0$ to orbit 2, that is, Id(2). Here $a = |\tilde{a}| \in (0, 2^{-2/3})$, $\tilde{e} = 1$, that is, $e = 0$, $C \geq 3.174$, $T = 2\pi(N-1)^{-1}$, $\mathrm{Tr} = 2\cos T$.
- $\mathbf{K}_2$. The family $K_2^{\pm}$ from Id(2) to $E_2(-0)$ (orbit 5_1). Orbit 3, where the body P_3 collides with the body P_1, is located in it. After this, the family i consists of triple orbits with a retrograde motion. In particular, when $\tilde{e} = -1$, the family i contains orbit 4 which is a triply passed orbit of the family Ir, that is, h. After this orbit when $\tilde{e} > -1$, this piece of the family i will be the family E_2^- and, up to this orbit, it was the family E_2^+. Here, $T = 2\pi$ and $\mathrm{Tr} = 2$.
- $\mathbf{K}_3$. The family C_{12} with $N > 2$, that is, with $a < 2^{-2/3}$ from $E_2(-0)$ to $E_2(+0)$, that is, orbit 7_1. When $\tilde{e} = 0$, this piece contains the orbit 6_1 in which the body P_3 collides with the body P_1. When the family passes through the orbit 6_1, the triple retrograde orbits pass into single direct orbits ($\tilde{e} > 0$). Here $\mathrm{Tr} = +\infty$.
- $\mathbf{K}_4$. The family E_2^- from $E_2(+0)$ to Id(2) (orbit 8, which is identical to orbit 2). In this piece, $T = 2\pi$ and $\mathrm{Tr} = 2$.
- $\mathbf{K}_5$. The family Id from Id(2) to Id(3/2), that is, of the orbit 10; $T = 2\pi(N-1)^{-1}$, $\mathrm{Tr} = 2\cos T$.
- $\mathbf{K}_6$. The family $E_{3/2}^{+}$ from Id(3/2) to $E_{3/2}(-1)$, that is, of the orbit 11_2; $T = 4\pi$, $\mathrm{Tr} = 2$.
- $\mathbf{K}_7$. The family $C_{12} + B_1$ from $E_{3/2}(+1)$ to $E_{3/2}(-1)$, that is, of the orbit 15_2. Each orbit consists of two arc-solutions: one from the family C_{12} and the other from the family B_1. We shall subsequently denote the piece of the generating family of periodic solutions, each orbit of which consists of a single arc-solution of the family $C_{k,k+1}$

and l identical arc-solutions of the family B_1, by $C_{k,k+1} + lB_1$. Here, the value of the Jacobi constant C is the same in them. The orbit 12_2 includes the extremal orbit $C_{12}(1)$, where C reaches a maximum value. In orbit 12_2, the right-hand characteristic of the family i reaches a maximum with respect to $\tilde{a}$ and a minimum with respect to $\tilde{e}$ after which it returns back along the characteristic of family B_1, that is, it has a cusp. In orbit 13_2, the body P_3 collides with the body P_1 on an arc-solution from the family C_{12} while, in orbit 14_2, the body P_3 collides with the body P_1 in an arc-solution of the family B_1, after which the direct motion changes to a retrograde motion. Here, the trace Tr in orbit 11_2 falls from 2 to $-\infty$, it then jumps up to $+\infty$ and remains at this level up to orbit 12_2 where it falls to $-\infty$ and remains the same up to the end of this piece.

K$_8$. The family $E^{\pm}_{3/2}$ from $E_{3/2}(-1)$ to $E_{3/2}(-0)$, that is, of the orbit 17_1; $T=4\pi$, Tr = 2. Orbit 16 is a quintuple circular orbit from the family Ir, that is, h. After it, the family $E^{+}_{3/2}$ passes into the family $E^{-}_{3/2}$.

K$_9$. The family C_{23} from $E_{3/2}(-0)$ to $E_{3/2}(+0)$, that is, of the orbit 19_1. In orbit 18, there is a collision between the body P_3 and the body P_1 and the direction of the motion changes. Here Tr = $+\infty$.

K$_{10}$. The family $E^{-}_{3/2}$ from $E_{3/2}(+0)$ to Id(3/2) (orbit 20 is identical to orbit 10); $T=4\pi$, Tr = 2.

K$_{11}$. The family Id from Id(3/2) to Id(4/3) (orbit 22); $T=2\pi(N-1)^{-1}$, Tr = 2cos$T$.

K$_{12}$. The family $E^{+}_{4/3}$ from Id(4.3) to $E_{4/3}(+1)$ (orbit 23_2); $T=6\pi$, Tr = 2.

K$_{13}$. The family $C_{23}+B_1$ from $E_{4/3}(+1)$ to $E_{4/3}(-1)$ (orbit 27_2). The orbit 24_2 includes the extremal orbit $C_{23}(1)$. In orbit 24_2, the right-hand characteristic (going along the characteristic of the family B_1) has a cusp. Orbits 25_2 and 26_2 are collision orbits. Here, the trace Tr in orbit 24_2 falls from 2 to $-\infty$ and then jumps to $+\infty$. In orbit 24_2, it jumps to $-\infty$ and remains so up to the end of this piece.

K$_{14}$. The family $E^{+}_{4/3}$ from $E_{4/3}(-1)$ to $E_{4/3}(-0)$ (the orbit 29_1) including the sevenfold orbit Ir(4/3) (orbit 28); $T=6\pi$, Tr = 2.

K$_{15}$. The family C_{34} from $E_{4/3}(-0)$ to $E_{3/2}(-1)$ (orbit 30_3); Tr = $+\infty$.

K$_{16}$. The family $C_{12}+2B_1$ from $E_{3/2}(-1)$ to $E_{3/2}(+1)$ (orbit 34_3), including the collision orbits 30_3 and 31_3 and the extremal orbit 33_3. Here, Tr = $+\infty$ up to orbit 33_3, where it jumps to $-\infty$ while, in orbit 34_3, it jumps to $+\infty$.

K$_{17}$. The family C_{34} from $E_{3/2}(+1)$ to $E_{4/3}(+0)$ (orbit 35_1); Tr = $+\infty$.

K$_{18}$. The family $E^{-}_{4/3}$ from $E_{4/3}(+0)$ to Id(4/3) (orbit 36 which is identical to orbit 22); $T=6\pi$, Tr = 2.

K$_{19}$. The family Id from Id(4/3) to Id(5.4) (orbit 38; $T=2\pi/(N-1)$, Tr = 2cos$T$.

K$_{20}$. The family $E^{+}_{5/4}$ from Id(5/4) to $E_{5/4}(+1)$ (orbit 39_2); $T=8\pi$, Tr = 2.

A more complete description[47] of the initial segment of the generating family and a description of the whole of this family are available. The point is that the initial segment which has already been described consists of cycles, each of which consists of a single piece of Id$_p$ and several pieces of the families $E^{\pm}_{(p+1)/p}$ and S. In this case, the cycle finishes in the same finite orbit of the piece Id$_p$ from which the piece of the family $E^{\pm}_{(p+1)/p}$ departed. The first cycle includes the pieces $\mathbf{K}_1-\mathbf{K}_4$. The second cycle consists of the pieces $\mathbf{K}_5-\mathbf{K}_{10}$ and the third cycle consists of the pieces $\mathbf{K}_{11}-\mathbf{K}_{18}$. The pieces $\mathbf{K}_{19}$ and $\mathbf{K}_{20}$ form the beginning of the fourth cycle. The whole of the generating family i consists of an infinite number of such cycles, the structure of which becomes more complex as the number p increases. In particular, all cycles, starting from the second cycle, have segments of the right-hand characteristic passing through the characteristic of the family B_1 and segments going along the curve P_2^{**} corresponding to the body P_2. These segments have a zig-zag structure which is shown schematically in Fig. 9 for segments passing through the characteristic of the family B_1. The number n of changes in the directions of the characteristics (zig-zags) is plotted along the ordinate axis in Fig. 9.

We will now consider the evolution of the family i as μ increases from zero. This is more conveniently done separately for each pair of cycles.

6.2. *Evolution of the first and second cycles*

The calculated fragments of the right characteristic of the family i when $\mu=\mu_J=9.5388\cdot10^{-4}$, $\mu=2\cdot10^{-3}$, $\mu=3\cdot10^{-3}$ are shown in Fig. 10, *a - c* respectively. It is clear that the first zig-zag of the characteristic as μ increases descends and then bends to the left and approaches the left-hand lower part of this fragment. When $\mu=\mu'_1\approx4.1313\cdot10^{-3}$, both parts of the characteristic are encountered (Fig. 10, *d*) and bifurcation occurs. When $\mu>\mu'_1$, a closed family is formed which we denote by i_1, and its characteristics are the closed curves shown in Fig. 11,

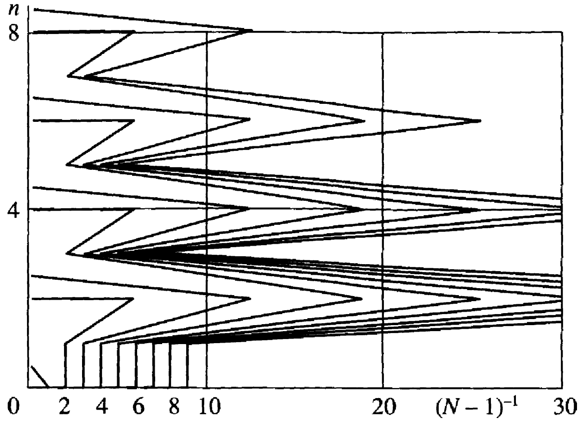


Fig. 9.

a for $\mu = \mu_M = 1.2155 \cdot 10^{-2}$, $\mu = 2.3 \cdot 10^{-2}$. As μ increases, the characteristics of the family i_1 decrease in size and, when $\mu = \mu''_1 \approx 3.66863 \cdot 10^{-2}$, the family i_1 contracts into the single orbit shown by the point in Fig. 11, a. Consequently, the family i_1 only exists in the interval $\mu \in [\mu'_1, \mu''_1]$. The unclosed characteristics correspond to $\mu = 5 \times 10^{-3}$ in Fig. 11, a. It is seen that, compared with Fig. 10, d, just one further bifurcation has occurred between the unclosed characteristics in the interval $\mu \in (\mu'_1, 5 \cdot 10^{-3})$.

6.3. Evolution of the second and third cycles

The calculated characteristics of parts of the second and third cycles when $\mu = 5 \times 10^{-4}$ are shown in Fig. 11,b. The evolution of the zig-zag third cycle, which is analogous to the evolution of the zig-zag of the second cycle can be seen. When $\mu = \mu'_2 \approx 6.61705 \cdot 10^{-4}$, bifurcation takes place, which is shown in Fig. 11,c. When $\mu > \mu'_2$, a closed family is formed which we denote by i_2. As μ increases, the characteristics of the family i_2 decrease in size; they are shown in

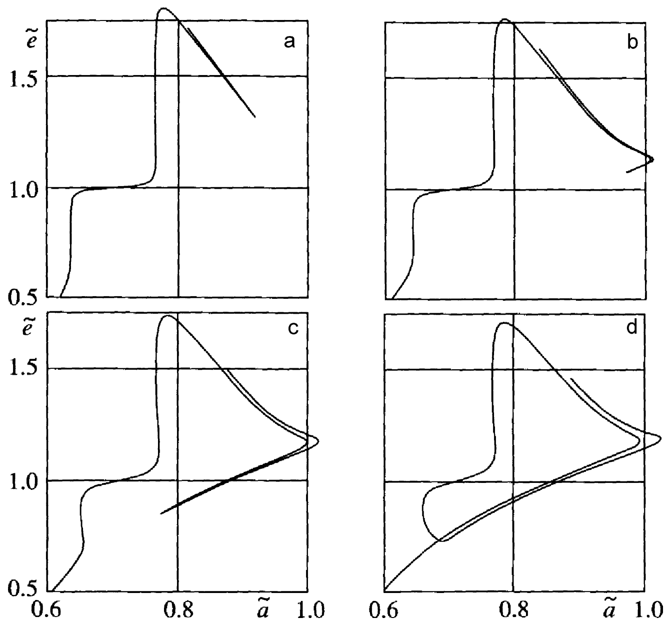


Fig. 10.

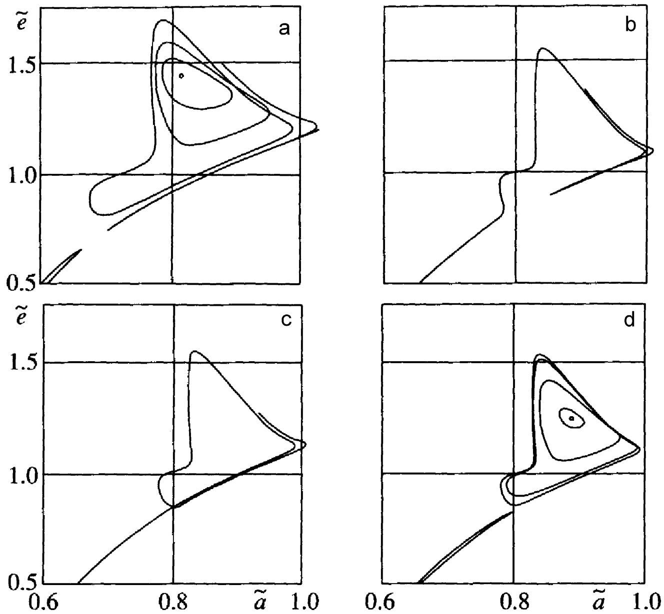


Fig. 11.

Fig. 11, d for $\mu = 7 \cdot 10^{-4}$, $\mu = \mu_J$, $\mu = 2.5 \cdot 10^{-3}$, $\mu = 5 \cdot 10^{-3}$ and, finally, when $\mu = \mu''_2 \approx 5.27272 \cdot 10^{-3}$, the family i_2 contracts into a single orbit, shown by the dot in Fig. 11, d. The unclosed characteristic below to the left in Fig. 11, d corresponds to $\mu = 7 \times 10^{-4}$. Consequently, the family i_2 only exists in the interval $\mu \in [\mu'_2, \mu''_2]$.

A closed family i_3 is formed in a similar manner from the third and fourth cycles when $\mu = \mu'_3 \approx 2.15292 \cdot 10^{-4}$, which exists up to $\mu = \mu''_3 \approx 1.88241 \cdot 10^{-3}$. Hence, the family i_3 exists when $\mu \in [\mu'_3, \mu''_3]$

The family i_4, which arises from the fourth and fifth cycles when $\mu = \mu'_4 \approx 9.54305 \cdot 10^{-5}$ and terminates when $\mu = \mu''_4 \approx 8.86552 \cdot 10^{-4}$, was also calculated.

6.4. Generalizations

6.4.1. Hypothesis

Two sequences μ'_k, μ''_k, $\mu'_k < \mu''_k$, $k = 1, 2, \ldots$, which decrease monotonically to zero, exist such that, when μ increases, closed families i_k are separated from the family i, which only exist in the intervals $\mu \in [\mu'_k, \mu''_k]$.

The initial parts of the sequences $\{\mu'_k\}$, $\{\mu''_k\}$ are shown in Table 3, where the empirical asymptotic forms of their normalized values are also indicated.

The families i_2 and i_3 have been calculated[10,11] when $\mu = \mu_J$, which corresponds to the Sun - Jupiter case. The characteristic of the family i_2 is shown in Fig. 11, d. The closed locally multiple families associated with them have been calculated in Ref. 12. When $\mu = \mu_M \approx 1.2155092 \cdot 10^{-2}$, which corresponds to the Earth - Moon case, there is a closed family i_1 (Fig. 11,a) which was not indicated when calculating the family i for μ_M.[15]

Table 3

k	μ'_k	μ''_k	μ'_k/μ'_1	μ''_k/μ''_1	$k^{-8/3}$
1	$4.131 \cdot 10^{-3}$	$3.669 \cdot 10^{-2}$	1	1	1
2	$6.617 \cdot 10^{-4}$	$5.273 \cdot 10^{-3}$	0.160	0.144	0.157
3	$2.153 \cdot 10^{-4}$	$1.882 \cdot 10^{-3}$	0.052	0.051	0.053
4	$9.543 \cdot 10^{-4}$	$8.866 \cdot 10^{-4}$	0.023	0.024	0.025

6.5. External annulus of almost circular orbits

The family i describes all of the almost circular periodic orbits of the direct motion with $a>1$, that is, from an internal annulus with respect to the body P_2. The periodic orbits of the direct motion from an external annulus to the body P_2 do not belong to a single family and are distributed in a denumerable set of families.

The families of periodic solutions, including the pieces

$$\mathrm{Id}_p = \left\{\frac{p}{p-1} > N > \frac{p+1}{p}\right\}$$

of the perturbed family Id of circular orbits with a direct sidereal motion for $p=-2, -3, \ldots, -7$ were calculated for $\mu=5.178\times10^{-5}$. The characteristics of the calculated families in the plane $\tilde{a}, \tilde{e}$ consist of horizontal fragments corresponding to the families Id_p, vertical fragments corresponding to the families $E_{p/(p-1)}$ (their bifurcations agree with Fig. 5) and inclined fragments going along the characteristics of the families A_i and B_j in Fig. 4, that is, which are located close to the characteristics of the generating families. An exception to this rule is the last family: it is closed, the value of μ indicated for it is not small and is already sufficient for self-bifurcation (as in the case of the family i)

7. Horseshoe-shaped orbits and orbits in the form of tadpoles

7.1. The neighbourhood of fixed points (Ref. 2, Ch. VIII, § 5; Ref. 28 § 2)

The synodic two-body problem for P_1 and P_3 has a one-parameter family of fixed points

$$x_1^2 + x_2^2 = 1, \quad y_1 = -x_2, \quad y_2 = x_1 \tag{7.1}$$

that is, $a=1$, $e=0$, $n=1$. In the neighbourhood of this family, we introduce the local coordinates y, z_1, z_2, z_3 as follows (Ref. 2, Ch. VIII, § 3). We start from the Delaunay elements

$$L = \sqrt{a}, \quad G = \sqrt{a(1-e^2)}, \quad l = Nt, \quad g = \tilde{\omega} - t$$

where $\tilde{\omega}$ is the argument of the pericentre. The system of canonical coordinates

$$L, \rho_2 = L - G, \quad y = l + g, \quad \varphi_2 = -g$$

is called the *first system of Poincaré elements*. Finally, the system of canonical coordinates

$$L, z_2 = \sqrt{2\rho_2}\cos\varphi_2, \quad y, \quad z_3 = \sqrt{2\rho_2}\sin\varphi_2$$

is called the *second system of Poincaré elements*.

The family (7.1) is isolated by the equalities $L=1$, $z_2=z_3=0$. The coordinates

$$z_1 = L - 1, \quad z_2, \quad y, \quad z_3 \tag{7.2}$$

will be local for the family (7.1). Hamiltonian function (1.2) has the form

$$H = H_0 + \mu R$$

$$H_0 = \rho_2 - z_1 - \frac{1}{2}(1+z_1)^{-2} = -\frac{1}{2} + \rho_2 - \frac{3}{2}z_1^2 - \frac{1}{2}\sum_{k=3}^{\infty}(k+1)(-z_1)^k \tag{7.3}$$

$$R = r^{-1} + r\cosh - (1 - 2r\cosh + r^2)^{-1/2}$$

(r and h are the polar coordinates of the body P_3 in the plane x_1, x_2). Here, $\rho_2 = z_2^2 + z_3^2$.

The function R is expanded in a convergent series of the form

$$R = \sum R_{mnkl} z_1^m z_2^k z_3^l \exp(iny)$$

with integers m, n, k and l and non-negative m, k and l.

According to calculations (Ref. 2, Ch. VII, Section. 5.B), we have

$$\beta_{000}(y) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} R_{0n00}\exp(iny) = 1 + \cos y - [2 - 2\cos y]^{-1/2} \tag{7.4}$$

In the coordinates (7.2), the family of fixed points is given by the equations

$$z_1 = z_2 = z_3 = 0$$

and the coordinate y is the angle of a fixed point in the plane x_1, x_2. According to formula (7.4), there is a singular perturbation when $y = 0$ as this point falls in the body P_2. Consequently, there is a broken circle in the restricted problem when $\mu = 0$, that is, an interval $y \in (0, 2\pi)$ of fixed points.

It has been shown (Ref. 2, Ch. VIII, Section. 3.B) that only three of them are generating points:

$$y_3^0 = \pi, \quad y_4^0 = \pi/3, \quad y_5^0 = 2\pi - \pi/3$$

The three fixed points L_3, L_4, L_5, which are the Lagrange solutions, correspond to them. A single generating family of periodic solutions: E_1^-, G_1^+ and G_1^- appears from each such point respectively. All the solutions of the generating families have a period $T = 2\pi$ and a trace $\mathrm{Tr} = 2$. When $\mu > 0$

$$T = 2\pi(1 + \dot{\tilde{\omega}}_1\mu + \phi(\mu^2)), \quad \mathrm{Tr} = 2 + \mathrm{Tr}_1\mu + \phi(\mu^2)$$

The values of $\tilde{\omega}$ and Tr_1 have been presented for the families G_1 and E_1^- (Ref. 2, Appendix; Tables 1 and 2). When $\mu = 0$, the families $G_1^{\pm}$ are intersected by the family E_1^- when $e = 0.917$ (Ref. 2, Fig. 17).

We will now study the periodic solution of the restricted problem for small $\mu > 0$, z_1, z_2, z_3 when $y \in (0, 2\pi)$. We shall assume that $z_i = O(\sqrt{\mu})$ and select the terms in Hamiltonian function (7.3) up to $O(\mu)$ inclusive. This first approximation to the Hamiltonian function and the corresponding Hamiltonian system have the form

$$\hat{H} = -\frac{1}{2} + \rho_2 - \frac{3}{2}z_1^2 + \mu\beta_{000}(y) \tag{7.5}$$

$$\dot{\rho}_2 = 0, \quad \dot{\varphi}_2 = -1, \quad \dot{z}_1 = \mu\frac{d\beta_{000}}{dy}, \quad \dot{y} = 3z_1 \tag{7.6}$$

We make the substitution

$$z_1 = \mu^{1/2}z, \quad \rho_2 = \mu\rho, \quad t = \mu^{-1/2}\tau, \quad \varphi_2 = \varphi \tag{7.7}$$

Not being canonical, that is, not retaining the Hamiltonian character of the whole system, it transfers system (7.6) the system

$$\rho' = 0, \quad \varphi' = -\mu^{-1/2} \tag{7.8}$$

$$y' = -\frac{\partial H}{\partial z}, \quad z' = \frac{\partial H}{\partial y}; \quad H = -\frac{3}{2}z^2 + \beta_{000}(y) \stackrel{\text{def}}{=} -\frac{3}{2}z^2 + \cos y - (2 - 2\cos y)^{-1/2} \tag{7.9}$$

where a prime denotes differentiation with respect to τ. Consequently, there is a first integral $\rho = \text{const}$ in the case of subsystem (7.8). The integral curves of subsystem (7.9), that is, the contour lines of the function H, are shown in Fig. 12. There are three types of orbits: 1) the librations about the point L_4 or the point L_5 are "tadpoles", 2) asymptotic to the point L_3, 3) librations of the asymptotic solutions in a "figure of eight" with horseshoe-shaped integral curves.

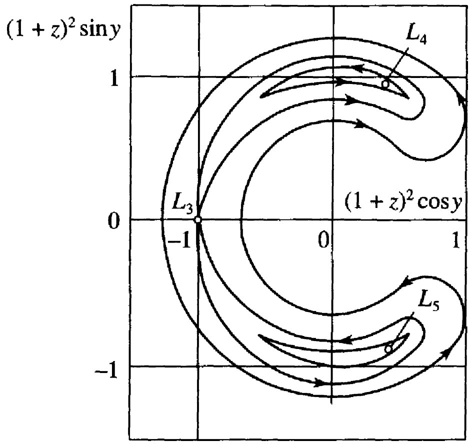


Fig. 12.

7.2. Periodic solutions of system (7.8), (7.9)

With the exception of the fixed points and the asymptotic solutions, all the remaining solutions of subsystem (7.9) are periodic. We will now find their periods. Subsystem (7.9) has the form

$$y' = 3z, \quad z' = \sin y(-1 + (2 - 2\cos y)^{-3/2}) \tag{7.10}$$

We put $v^2 = 2 - 2\cos y$. Then, $|v| \leq 2$ and

$$\sin y = v\left(1 - \frac{v^2}{4}\right)^{1/2}, \quad y' = v'\left(1 - \frac{v^2}{4}\right)^{-1/2}, \quad H = -\frac{3}{2}z^2 + 1 - \frac{v^2}{2} - \frac{1}{v} \tag{7.11}$$

The contour lines of the function H are shown in Fig. 13 in $z, v \geq 0$ coordinates. The point (0, 1) is a fixed point which corresponds to the point L_4. The point L_3 corresponds to the point (0, 2). At this point and on the asymptotic curve, $H = -3/2$. The asymptotic curve intersects the v axis when $v = \sqrt{2} - 1$. In the contour lines of the function $H = \text{const}$ we have

$$z^2 = \frac{1}{3}\left(2 - 2H - v^2 - \frac{2}{v}\right)$$

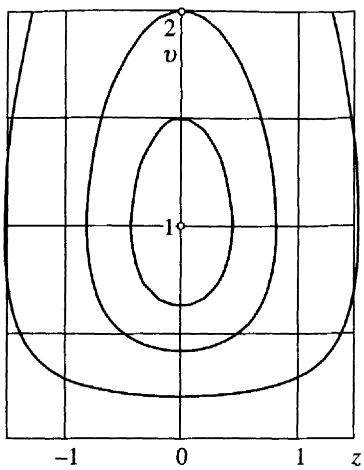


Fig. 13.

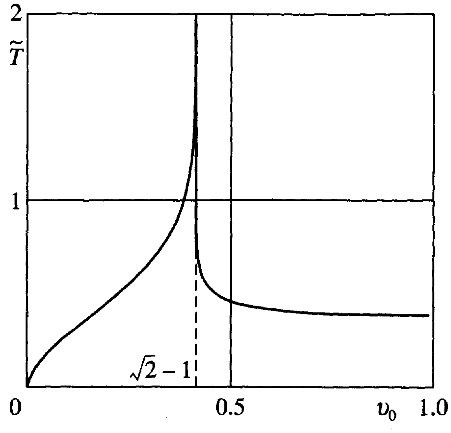


Fig. 14.

and, therefore, from the first equation of system (7.10) and equalities (7.11) we obtain

$$\frac{2}{\sqrt{3}}\upsilon^{1/2}[(4-\upsilon^2)(2\upsilon-2H\upsilon-\upsilon^3-2)]^{-1/2}d\upsilon = d\tau \tag{7.12}$$

For the point $z=0$, $\upsilon=\upsilon_0$, we find

$$H = H_0 = 1-\upsilon_0^2/2-\upsilon_0^{-1} \tag{7.13}$$

On the υ axis, the curve $H=H_0=\text{const}$ has a further two points: the roots of the equation $2-\upsilon_0^2\upsilon-\upsilon_0\upsilon^2=0$ and, of these, only one is positive

$$\upsilon_1 = \frac{1}{2}\left[\left(\upsilon_0^2+\frac{8}{\upsilon_0}\right)^{1/2}-\upsilon_0\right]$$

We will integrate Eq. (7.12) along a contour line from $\upsilon_0\in(0, 1)$ to υ_2, where $\upsilon_2=2$ if $\upsilon_0\in(0, \sqrt{2}-1)$ and $\upsilon_2=\upsilon_1$ if $\upsilon_0\in(\sqrt{2}-1, 1)$. When account is taken of formula (7.13), we obtain

$$I(\upsilon_0) = \frac{2}{\sqrt{3}}\int_{\upsilon_0}^{\upsilon_2}(\upsilon_0\upsilon)^{1/2}[(4-\upsilon^2)(\upsilon-\upsilon_0)(2-\upsilon_0^2\upsilon-\upsilon_0\upsilon^2)]^{-1/2}d\upsilon \tag{7.14}$$

In the case of horseshoe-shaped orbits $\upsilon_0\in(0, \sqrt{2}-1)$ and the period $T(\upsilon_0)=4I(\upsilon_0)$. For orbits in the shape of tadpoles $\upsilon_0\in(\sqrt{2}-1, 1)$ and the period $T(\upsilon_0)=2I(\upsilon_0)$. A graph of the function $\tilde{T}(\upsilon_0)\overset{\text{def}}{=}T(\upsilon_0)/(2\pi)$ is shown in Fig. 14. Here, $\tilde{T}(\upsilon_0)\to 2/(3\sqrt{3})\approx 0.385$ if $\upsilon_0\to 1$. When $\upsilon_0\to\sqrt{2}-1$ from both sides, $\tilde{T}(\upsilon_0)\to+\infty$.

The solutions of system (7.8), (7.9) are situated in the invariant tori

$$\rho = \rho_0 = \text{const}\neq 0, \quad H = H_0 = \text{const}$$

and in the invariant manifold $\rho=0$. The frequency of the periodic solutions in this manifold is equal to $\omega_1=1/\tilde{T}(\upsilon_0)$ and the frequency of the external rotation through an angle φ is equal to $\omega_2=-1/\sqrt{\mu}$.

We put

$$n = -\frac{\omega_2}{\omega_1} = \frac{\tilde{T}(\upsilon_0)}{\sqrt{\mu}} \tag{7.15}$$

7.3. Local families of periodic solutions

We now return to system (7.6). The structure of its solutions is the same as in the case of system (7.8), (7.9). In particular, it has the invariant manifold $\rho_2 = 0$ filled with periodic solutions with the frequency ratio (7.15). The complete system (7.3) is a perturbation of system (7.6). Under such perturbations, bifurcations of the families of periodic solutions arise for those solutions from the manifold $\rho_2 = 0$ for which the frequency ratio (7.15) is an integer. This occurs for the values of v_0 for which

$$\tilde{T}(v_0) = n\sqrt{\mu} \tag{7.16}$$

It can be seen from Fig. 14 that, for any $\mu \in (0, 1/2)$, unique values

$$v_0 = v_0^{(n)} \in (0, \sqrt{2} - 1), \quad v_0 = \tilde{v}_0^{(n)} \in (\sqrt{2} - 1, 1)$$

exist for which equality (7.16) with $n > 0$ and $n > 2/(3\sqrt{3\mu})$ are satisfied respectively, and there is therefore a bifurcation horseshoe-shaped orbit for each natural n and a bifurcation orbit in the shape of a tadpole close to the point L_4 but only for each integer $n > 2/(3\sqrt{3\mu})$ (and, similarly close to the point L_5).

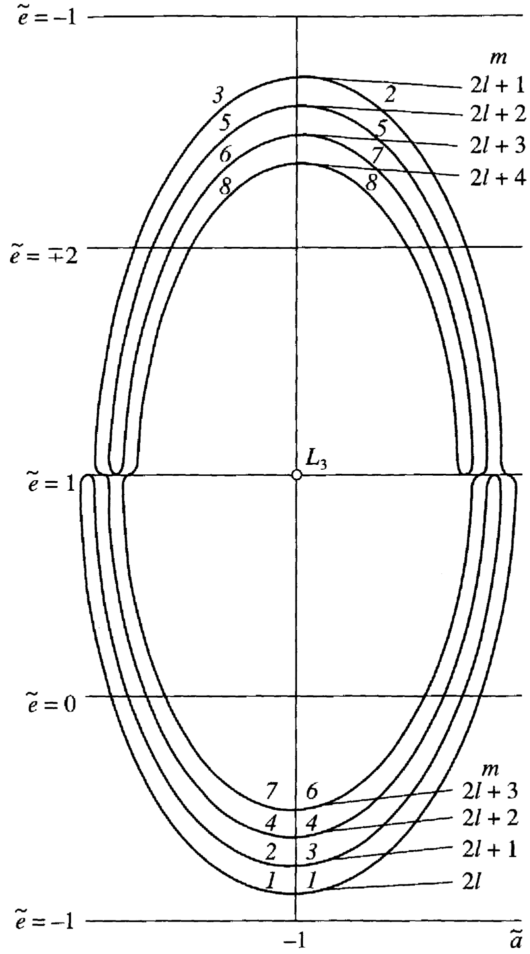
7.4. The global structure of families with horseshoe-shaped orbits

Periodic solutions with horseshoe-shaped orbits are symmetric and they therefore intersect the plane of symmetry II, and the characteristics of the families of such solutions form curves in the II plane. The arrangement of these characteristics for small $\tilde{e} - 1$ is shown schematically in Fig. 15, which is obtained if the results in Ref. 48 are plotted in $\tilde{a}$, $\tilde{e}$ coordinates. We call families, with characteristics in $\tilde{a}$, $\tilde{e}$ coordinates containing a horizontal segment $\tilde{e} \approx 1$, *basic* families and we denote them by FH. If these families are continued, they intersect with the family E_1^- as locally multiple families.

Taking into account the local results of Subsection 7.3 and Ref. 48 and the global results of, Refs. 49,50 we obtain the arrangement of the characteristics of the natural basic families with horseshoe-shaped orbits FH shown in Fig. 15 in $\tilde{a}$, $\tilde{e}$ coordinates. Here, the strip $\tilde{e} \in [-1, -2]$ is put on top for continuity of the characteristics. The characteristics of two natural closed families are shown which periodically repeat themselves both within and outside the annulus, which has been depicted. The vertical segment is the characteristic of the family E_1^- which starts from the fixed point L_3. The numbers m at the positions of its intersection with the characteristics of the families FH indicate the local multiplicity of these families; they are located to the right. The number of a curve (from 1 to 8) is placed in each quadrant of Fig. 15 above each curve. The same numbers of the non-horizontal segments of the characteristics indicate the segments of the two characteristics corresponding to a single segment of a family. The segments of the characteristics with the numbers 1, 2, 3, 4 belong to a single family and those with the numbers 5, 6, 7, 8 to another family. All of this is in accord with the numerical results for horseshoe-shaped orbits.[4,5,48]

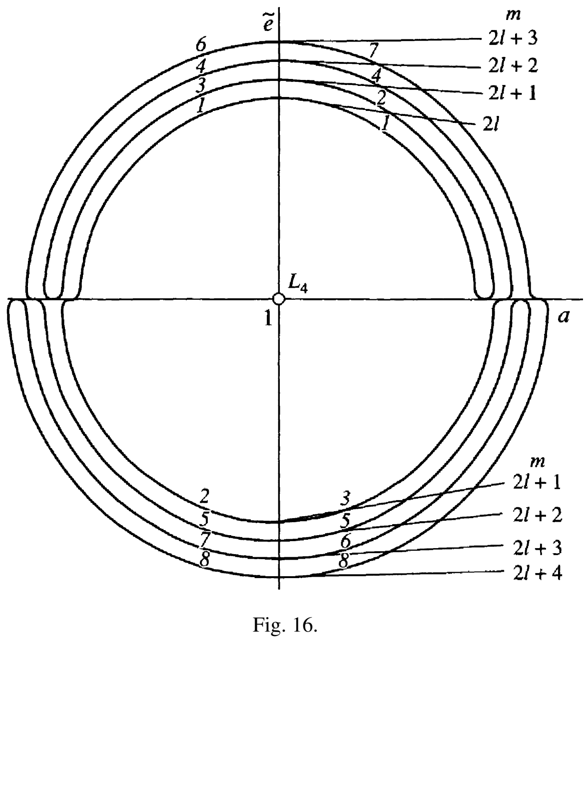
7.5. The global structure of families with orbits in the shape of tadpoles

In principle, the structure of these families is similar to the structure of the families of horseshoe orbits. The basic difference lies in the fact that the tadpole orbits are not symmetrical and it is impossible to depict the characteristics of their families in the plane of symmetry II. However, they can be depicted as curves in a three-dimensional section Γ of phase space (Ref. 2, Ch. III, Section 2.E) with a, θ, $\tilde{e}$ coordinates. Since each such family intersects with the family G_1 as a locally m-fold family, $2m$ characteristics correspond to it. Only two of these characteristics exist for small $|\tilde{e} - 1|$, and we call them the *principle* characteristics of the family. We will now consider the projections of the principle characteristics onto the half-plane a, $\tilde{e}$. As before, we shall call those families for which these projections contain a horizontal segment $\tilde{e} \approx 1$ basic families FT. The projections of the principal characteristics of the basic families FT are shown schematically in Fig. 16 which is similar to Fig. 15. The vertical segment in Fig. 16 is the projection of the characteristic of the family G_1, the point $a = \tilde{e} = 1$ is the projection of the fixed point L_4 and an increase in the multiplicity of the intersections of the families FT with the family G_1 in a reverse direction: from the inside to the outside.



ẽ = −1
m
2l + 1
2l + 2
2l + 3
2l + 4
ẽ = ∓2
L3
ẽ = 1
ẽ = 0
m
2l + 3
2l + 2
2l + 1
2l
ẽ = −1
−1
ã

Fig. 15.



ẽ
m
2l + 3
2l + 2
2l + 1
2l
L4
1
a
m
2l + 1
2l + 2
2l + 3
2l + 4

Fig. 16.

In traditional terminology, the family G_1 is the family of short-period solutions starting from the fixed point L_4, the horizontal segments of the characteristics in Fig. 16 correspond to the family of long-period solutions starting from the fixed point L_4 when $\mu > 0$ and the basic families FT themselves are the bridges of the resonance periodic solutions.[51]

Remark. The periodic solutions considered in Section 7 do not have generating solutions in the basic limit problem (nor in the remaining limit problems either). Although the families i_k, considered in Section 6 do have a generating family i, they are very different from it and they cannot be considered as just perturbations of the generating family. These examples show that the generating solutions of the basic limit problem do not suffice to describe the families of periodic solutions of the restricted problem for small μ.

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We wish to dedicate this paper to the 300-th anniversary of the birth of Leonhard Euler.

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